LOW SENSITIVITY DESIGN OF OPTIMAL FEEDBACK SYSTEMS FOR LONGITUDINAL CONTROL OF AUTOMATED TRANSIT VEHICLES

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CHAPTER 1

INTRODUCTION

Section 1.1 Motivation

Almost all new urban transportation systems involve the use of automatically controlled vehicles. [T2]*. Many new systems, such as Personal Rapid Transit, Dual Mode, and Automated Highways, are characterized by small automated transit vehicles travelling on the exclusive guideways. The number of passengers per vehicle is small and short headways are necessary for high capacity. This requires a versatile, efficient, and safe control system which maintains the proper spacing between vehicles on the guideways without causing passenger discomfort. ([G1], [H2]). The longitudinal control system is an essential part of the overall control system for automated transit vehicles. The longitudinal control system must be capable of closely following the acceleration-deceleration profiles as commanded by wayside computers during merging and demerging, maneuvering to avoid conflicts at intersections, pushing a failing vehicle, or stopping for an emergency. [G4]. The research reported on here was devoted to the design of such a longitudinal control system using modern control technology.

*Numbers in brackets refer to references in the Bibliography.

Section 1.2 Survey of Related Works

Two control philosophies for the longitudinal control of small automated transit vehicles have evolved. The first is the internal-reference or vehicle-following concept. ([A4], [A5], [B1], [G3], [L5], [M1], [P1], and [P3]). In systems based on this concept, each vehicle receives information directly from other vehicles on the guideway or merging ramps. The control decisions are based on the relative position and absolute velocity of the vehicles.

The second control philosophy is the external-reference or moving-slot concept. ([B3], [G1], [G4], [L6], [M2], [S3], [W1], and [W2]). In systems based on this concept, the moving vehicle follows an alloted hypothetical slot moving along the guideway at the nominal line velocity. The slot lengths are uniform and are equal to the length of the vehicle plus the minimum allowable nose to tail separation between adjacent vehicles. Vehicles do not communicate directly with one another. The control decisions are based on the position of the vehicle with respect to the guideway and velocity of the vehicle compared with the nominal line velocity.

A number of investigators have concluded that external-reference systems are superior to internal-reference systems in high-capacity transit systems in which vehicles must merge and demerge from off-line stations and maneuver from one slot to another in order to resolve conflicts at merges or interchanges. ([B3], [G1], [M2], [W1], and [W2]). Furthermore, the external-reference systems are superior for the following reasons:

(1) Communication problems are reduced since intervehicular communication is not necessary.

(2) Merging, maneuvering, and emergency operations are easily implemented.

(3) Possible shock-wave type instabilities in a string of vehicles are avoided.

The design of external-reference control systems has received a great deal of attention recently. ([G1], [G4], [W1], and [W2]). Wilkie [W2] derived a P-I-D type controller (the control input is proportional to the headway error, the integral of the headway error, and the derivative of the headway error) for normal operation in the continuous case by using optimal regulator theory. Whitney and Tomizuka [W2] derived a similar P-I-D controller which gave good performance for normal vehicle control in the sampled-data case by using the classical control techniques. Garrard and Kornhauser [G1] studied the problem including the dynamics of propulsion system. Their results showed that the control input to the propulsion system is P-I-D type and also proportional to the propulsive force. It should be noted that the steady-state errors due to constant biases in the measurements of velocity errors, acceleration errors, and the rate of change of propulsive force are zero in all of these systems. In practice, it is inconvenient and expensive to measure all the state variables. Garrard and Kornhauser [G4] introduced the theory of observers to estimate the unmeasured state variables. However, the random nature of the environmental disturbances, imperfect and noisy sensing, and the rolling resistance acting on the moving vehicle were not considered in any of these studies. Moreever, no studies of the sensitivity of the dynamic response of the vehicle to parameter changes have appeared in the literature.

There are several variable parameters in the differential equations which describe the longitudinal motion of a transit vehicle. The variable parameters considered in this study are the magnitude of the wind gust and the mass of vehicle, passengers, and goods on board. The effects of the vehicle mass and the wind gust are important since the vehicle is small and headways are short. The objective of this study is to provide a methodology for the design of low sensitivity longitudinal controllers for small automated transit vehicles such that the dynamic response of the vehicle will remain essentially invariant as the vehicle parameters vary. Furthermore, the random nature of the environmental disturbances, imperfect and noisy sensing, and rolling resistance were considered during the study.

Section 1.3 Summary and Structure of Thesis

In Chapter 2, a detailed mathematical model of the longitudinal motion of automated transit vehicles in an external-reference system is presented. The model is nondimensionalized so that the results are applicable to a variety of systems. The appropriate choice of state variables is discussed. The performance index is selected such as to give tight vehicle control as well as a comfortable ride. Formulations of the optimal design and the low sensitivity design problems are presented.

In Chapter 3, the theory of the low sensitivity state feedback design of a class of linear time-invariant system is reported. The low sensitivity design problem as formulated in Chapter 2 is a special case of the general problem formulated in this chapter. The optimization problem is converted into matrix differential equation form. The necessary conditions for optimality are obtained through the use of matrix minimum principle.

In Chapter 4, the study results of the longitudinal control problem in the presence of the disturbance environment and sensor errors are presented. Modern control and estimation theories were combined and applied to the design of a longitudinal control system for automated transit vehicles. The optimal estimator which minimizes the mean square error of estimation turned out to be a Kalman-Bucy type filter. The optimal longitudinal controller was shown to be the same as the deterministic controller discussed in Chapter 3.

In Chapter 5, a specific example contains details of the design of a high-capacity PRT system. The selection of control gains is discussed, and simulation results for normal operation and nonlinear profile-following operation are shown. The effects of eliminating the velocity sensor were investigated. The control system performance is studied as a function of the uncertainty of the measurements associated with the sensors. The amount of sensor error which is tolerable is an important economic factor as the cost of the control system is proportional to the quality of sensors required. Comparisons are made between the control system using stochastic estimators and the control system using deterministic observers.

In Chapter 6, conclusions of the study, contributions, and suggestions for future research in this area are discussed.

CHAPTER 2

VEHICLE DYNAMICS AND LONGITUDINAL CONTROL PROBLEM FORMULATION FOR AN EXTERNAL REFERENCE SYSTEM

Section 2.1 Introduction

In this chapter, the mathematical model which will be used in the design of the longitudinal control system is presented. In Section 2.2 the vehicle dynamics are carefully defined. A set of nondimensional system equations which have two variable parameters in the system matrix is described in Section 2.3. These parameters are the dimensionless wind gust velocity and the dimensionless mass. The synthesis of the optimal feedback longitudinal control system study is reported on Section 2.4. The appropriate choice of the state variables is discussed in Section 2.4.1. The performance index for the control system was chosen such that it gave tight headway control as well as a comfortable ride. The optimal design based on average values of the headwind velocity and vehicle mass is formulated in Section 2.4.2. A low sensitivity design problem formulation which takes into account variations in dynamic response due to changes in headwind velocity and vehicle mass is discussed in Section 2.4.3.

Section 2.2 Vehicle Dynamics

The forces acting on a moving vehicle are as illustrated in Fig. 2.1. ([D1], [G1], and [W1]) The equation of motion is

$$M \frac{d^2x}{dt^2} = F - F_a - F_r - Mg\sin\theta \tag{2.1}$$

where x = position of vehicle (measured in feet from a reference)

M = mass of vehicle, including passengers and goods on board (in slugs)

g = gravitational constant (32.2 ft/sec^2)

θ = slope of guideway (rad.)

The aerospace drag force F_a is generally of the form ([P1],[V1])

$$F_a = C_a \left(\frac{dx}{dt} + V_w\right)^2 \tag{2.2}$$

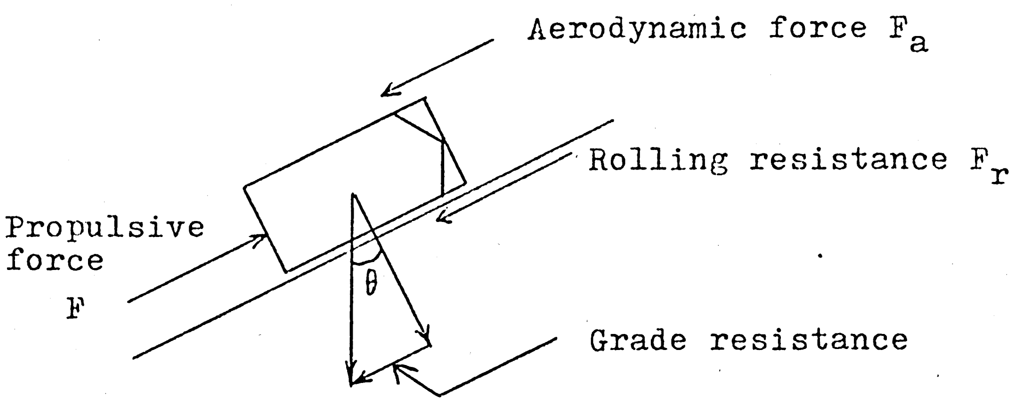


Fig.2.1 Forces on a moving vehicle

where C_a is a drag coefficient, assumed to be constant and V_w is the wind gust velocity, positive for head wind. Furthermore, the propulsion system is assumed to be governed by a first-order lag system ([G1])

$$\frac{dF}{dt} = -(\frac{1}{\tau})F + ki \tag{2.3}$$

where τ = time constant of the propulsion system

k = gain constant of the propulsion system

i = control input to the propulsion system

The rolling resistance, F_r, is composed of those forces inherent in the vehicle that tend to retard its motion. The formula for the rolling resistance is ([D1],[S3])

$$F_r = (C_s + C_r \frac{dx}{dt}) M \tag{2.4}$$

where C_s and C_r are constants.

For the external reference system, a command or desired position of the vehicle, x_c, is necessary. We will define the vehicle position error variable to be the position error with respect to this command position. That is.,

$$e \triangleq x - x_c \tag{2.5}$$

The equation of motion (2.1) may then be written in terms of the position variable as

$$\frac{d^2e}{dt^2} = \frac{F}{M} - \frac{C_a}{M}(\frac{de}{dt} + \frac{dx_c}{dt} + V_w)^2 - C_s - C_r(\frac{de}{dt} + \frac{dx_c}{dt}) - g\sin\theta - \frac{d^2x_c}{dt^2} \tag{2.6}$$

Section 2.3 Nondimensional System Model

For purpose of generality, the system equations will be nondimensionalized. This simplifies the system equations and makes the results applicable to a variety of systems. The nondimensional variables are defined as follows:

$y = \frac{e}{H}$ nondimensional position error

$y_c = \frac{x_c}{H}$ nondimensional commanded position

$I = \frac{T^2 ki}{M_o V_c}$ nondimensional control input to the propulsion system

$f = \frac{TF}{M_o V_c}$ nondimensional propulsive force

$w = \frac{V_w}{V_c}$ nondimensional headwind velocity

$m = \frac{M}{M_o}$ nondimensional mass of vehicle

$\sigma = \frac{t}{T}$ nondimensional time

$\cdot = \frac{d}{d\sigma}$ derivative with respect to nondimensional time

where H = nominal nose to nose distance between vehicles on the mainline guideway

T = nominal time headway between vehicles on the mainline guideway

V_c = command velocity on the mainline guideway, $H = V_c T$

M_o = the average mass of vehicle with passengers and goods on board

In terms of these nondimensional quantities, the vehicle dynamics equations are

$$\ddot{y} = \frac{f}{m} - \frac{1}{m}\left(\frac{C_a H}{M_o}\right)(\dot{y} + \dot{y}_c + w)^2 - \frac{T}{V_c} C_s - T C_r(\dot{y} + \dot{y}_c) - \frac{Tg\sin\theta}{V_c} - \ddot{y}_c \tag{2.7}$$

$$\dot{f} = -\frac{T}{\tau} f + I \tag{2.8}$$

During normal mainline operation, the vehicle will travel at nearly the commanded velocity so that a linearization of equations (2.7) and (2.8) is legitimate. Thus

$$\dot{y}_c = \left(\frac{1}{V_c}\right)\left(\frac{dx_c}{dt}\right) = 1 \tag{2.9}$$

$$\ddot{y}_c = \left(\frac{T}{V_c}\right)\left(\frac{d^2x_c}{dt^2}\right) = 0 \tag{2.10}$$

The resulting linearized equations are

$$\ddot{y} = -\left[\frac{a(1+w)}{m} + T C_r\right]\dot{y} + \frac{f}{m} - f_d \tag{2.11}$$

$$\dot{f} = -\left(\frac{T}{\tau}\right) f + I \tag{2.12}$$

where $a = \dfrac{2C_a H}{M_o}$

$$f_d = \frac{C_a H}{M}(1+w)^2 + \frac{Tg\sin\theta}{V_c} + \frac{T}{V_c} C_s + T C_r ,$$

is the nondimensional disturbance force.

Section 2.4 Synthesis of the Optimal Feedback Control System

The vehicle dynamics have been modeled by a set of linear differential equations (2.11) and (2.12). If the state variables are chosen properly, it is possible to use optimal control theory to design a feedback longitudinal controller which will keep position and velocity errors small without causing passengers discomfort. ([G1]) For purpose of control system design, the nondimensional disturbance force f_d is assumed to be constant.

Section 2.4.1 The State Variable

The appropriate state variables for the longitudinal control problem are position error, velocity error, acceleration error, and the rate of change of propulsive force. The reasons for this selection will be discussed in detail below.

The state variables are chosen as

$x_1 = y$ nondimensional position error

$x_2 = \dot{y}$ nondimensional velocity error

$x_3 = \ddot{y}$ nondimensional acceleration error

$x_4 = \dot{f}$ the rate of change of the nondimensional propulsive force

The control variable is $u = \dot{I}$, the rate of change of the nondimensional input to the propulsion system. Using these definitions, the vehicle dynamics can be written in vector-matrix form as

$$\dot{\underline{x}} = A\underline{x} + \underline{b}u \tag{2.13}$$

where $\underline{x}^T = (x_1 \quad x_2 \quad x_3 \quad x_4)$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a(1+w)}{m} - TC_r & \frac{1}{m} \\ 0 & 0 & 0 & -\frac{T}{\tau} \end{bmatrix}$$

$$\underline{b}^T = (0 \quad 0 \quad 0 \quad 1)$$

The control, $u = u(t)$; $t \geq 0$, will be selected in such a way that it drives the state vector $\underline{x}$ to zero. The state variables x_1 and x_2 represent the position and velocity errors and the reason for requiring of these quantities to be zero is clear. The state variables x_3 and x_4 represent the acceleration error and the rate of change of propulsive force, respectively. The acceleration error must be zero if the position and velocity errors are to remain zero; furthermore, in order to achieve zero acceleration error, the propulsive force must equal the disturbance force. Since the disturbance force is assumed to be constant, the propulsive force must also approach a constant value, and the derivative of the propulsive force, x_4, must approach zero. Also, since the drag coefficient C_a and the coefficient of rolling resistance C_r are small, the state variable x_4 is approximately equal to the nondimensional jerk, i.e.,

$$x_4 = \left(\frac{T^2}{V_c}\right)\left(\frac{\frac{d}{dt} F}{M_o}\right) \approx \left(\frac{T^2}{V_c}\right) \text{(actual jerk)}$$

$$= \text{nondimensional jerk} \qquad (2.14)$$

More precisely, the state variable x_4 serves as an upper bound of the nondimensional jerk. Hence, the selection of x_3 and x_4 as state variables will not only eliminate the steady-state errors due to constant disturbance forces, but also result in a comfortable ride. ([A6], [G1])

Section 2.4.2 The Performance Index and Optimal Design Problem

The system equations have been modeled in the standard notation of optimal control theory. In order to apply this theory a mathematical criterion for the measurement of system performance is necessary. Observe that there are two variable parameters w and m in system matrix (2.13). The dimensionless wind gust w and the dimensionless mass m are not known *a priori*; however, their average values will in general be zero and one, respectively. In the worst case, w is assumed to vary between -3 and +3, and m is assumed to vary between .5 and 1.5. A control system may be designed based on the average values of these parameters. Such a design will be called the OPTIMAL DESIGN. ([G1])
The system equations based on the average values of parameters are

$$\underline{\dot{x}} = A_o\underline{x} + \underline{b}u \tag{2.15}$$

where

$$A_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -a-TC_r & 1 \\ 0 & 0 & 0 & -\frac{T}{\tau} \end{bmatrix}$$

Note that the system described in (2.15) is completely controllable [L3], since

$$\text{rank}(\underline{b},\ A_o\underline{b},\ A_o^2\underline{b},\ A_o^3\underline{b}) = 4$$

The proposed performance index of the optimal design is in the quadratic form of the weighted sums of state variables and the control variable

$$J = \tfrac{1}{2} \int_0^\infty (\underline{x}^T Q_1 \underline{x} + r_1 u^2)\, dt \tag{2.16}$$

where Q_1 = diag. (q_1 q_2 q_3 q_4) the state weighting matrix and

r_1 is the control weighting factor

All the q's and r_1 are positive constants.

It is possible to fomulate other performance indices which include position and velocity errors as well as comfort of ride; however, the choice of a quadratic form as in (2.16) permits the optimal controller to be determined in a feedback form [u = u(x)] which is easy to implement. This is one of the few classes of optimal control problems in which the optimal feedback control can be found. ([L3],[A2])

The optimal feedback control problem is to determine the control, u, as a function of the state variables such that J is minimized as the state variables are driven to zero.

The performance index (2.16) penalizes not only large position and velocity errors, but also large acceleration and jerk errors. This will result in a vehicle control system in which tight control of position error and a comfortable ride are maintained. The control variable must also be included in the performance index for mathematical reasons.

The reasons for the choice of the control interval to be ($(0,\infty)$ are as follows:

(1) To insure the constant feedback gain matrix

(2) To make sure that the state variables will be driven to zero as time goes on.

The well-known optimal regulator theory shows that the solution of this optimization problem yields a unique optimal controller ([L3],[A2])

$$u = -G\underline{x} \tag{2.17}$$

where

(1) $G = r_1^{-1}\underline{b}^T K$ (2.18)

(2) K is the unique constant 4x4 positive-definite solution of the matrix Riccati equation

$$KA_o + A_o^T K - K\underline{b}r_1^{-1}\underline{b}^T K + Q_1 = 0 \qquad (2.19)$$

The vehicle dynamic response is the solution of the linear time-invariant homogeneous system

$$\dot{\underline{x}} = (A_o - \underline{b}G)\underline{x} \qquad x(0) \text{ given} \qquad (2.20)$$

From (2.17) the optimal control in terms of the actual error variables is

$$u = -\frac{G_{11}}{H}e - \frac{G_{12}}{V_c}\dot{e} - \frac{G_{13}T}{V_c}\ddot{e} - \frac{G_{14}T^2}{M_o V_c}\dot{F} \qquad (2.21)$$

The overall control system configuration is as shown in Fig. 2.2.

The optimal design is based on the average values of the vehicle parameters. In the sequel, it is desired to design longitudinal controllers such that the sensitivity of vehicle dynamic response to parameter variations is minimized. This design will be called the LOW SENSITIVITY DESIGN.

Section 2.4.3 The Low Sensitivity Design Problem

We will denote the A_o matrix in the Optimal Design problem as the nominal A matrix. We will denote the corresponding vehicle dynamic response as the nominal trajectory $\underline{x}(t;A_o)$, and the corresponding control as

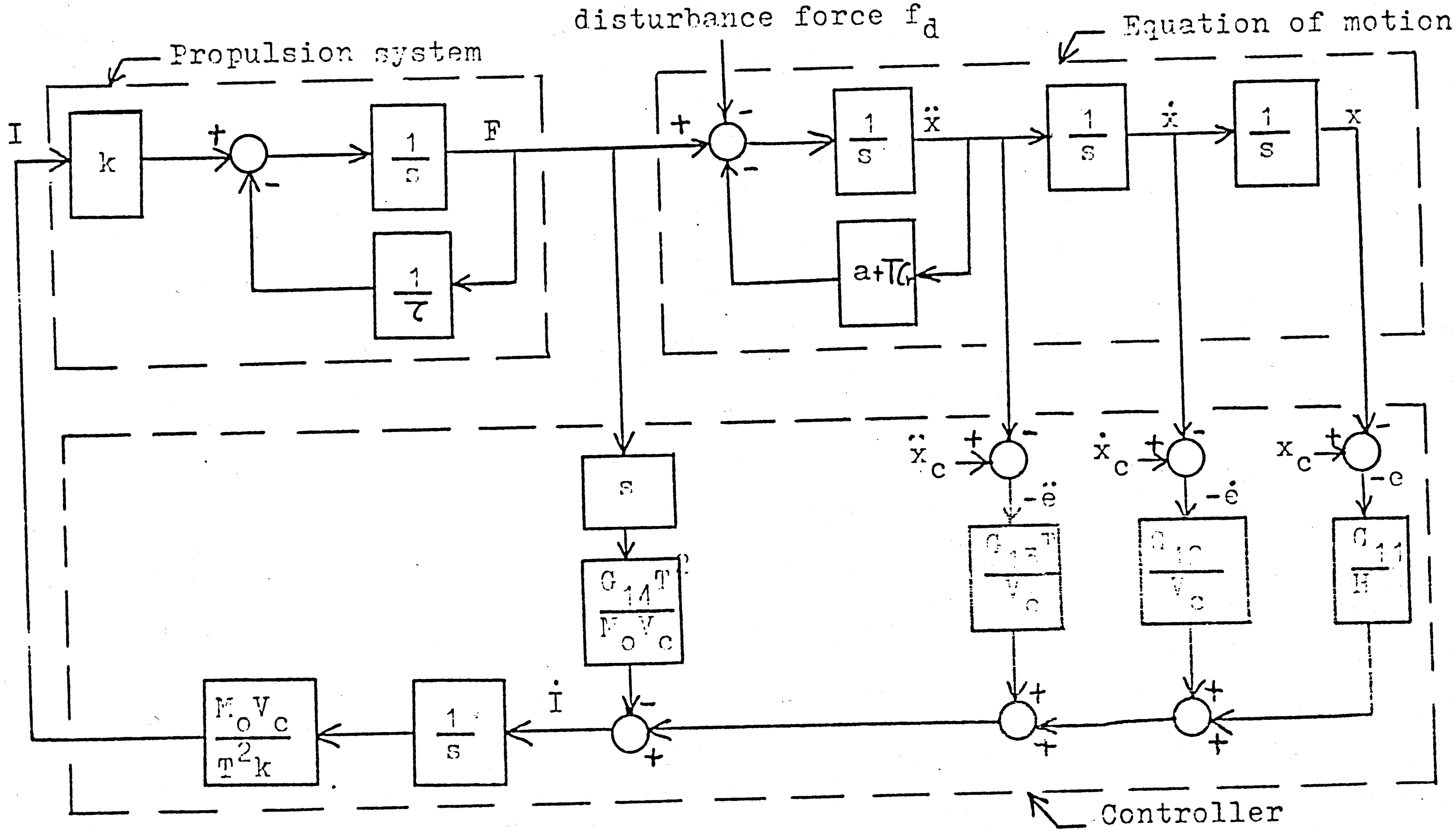


Fig.2.2 The overall control system configuration

$u(t;A_o)$. Define the trajectory sensitivity vector with respect to wind gust as

$$\underline{s}_w(t;A_o) \triangleq \left. \frac{\partial \underline{x}(t;A)}{\partial w} \right|_{(w,m)=(0,1)} \tag{2.22}$$

and define the trajectory sensitivity vector with respect to mass as

$$\underline{s}_m(t;A_o) \triangleq \left. \frac{\partial \underline{x}(t;A)}{\partial m} \right|_{(w,m)=(0,1)} \tag{2.23}$$

It is desired to design a feedback controller such that the trajectory $\underline{x}(t;A)$ will remain as nearly invariant as possible when the mass of the vehicle and the magnitude of the wind gust vary from their average values. To this end, the trajectory sensitivity vectors will be included in the performance index and, thus, penalizes large variations in the dynamic response of the vehicle due to variations in vehicle parameters. The new performance index is

$$J = \tfrac{1}{2} \int_0^\infty [\underline{x}^T(t;A_o)Q_1\underline{x}(t;A_o) + r_1u^2(t;A_o) + \underline{s}_w^T(t;A_o)W_1\underline{s}_w(t;A_o) + \underline{s}_m^T(t;A_o)M_1\underline{s}_m(t;A_o)]\, dt \tag{2.24}$$

where $Q_1 = \text{diag.}\ (q_1 \quad q_2 \quad q_3 \quad q_4)$ the state weighting matrix

r_1 is the control weighting factor

$W_1 = \text{diag.}\ (w_1 \quad w_2 \quad w_3 \quad w_4)$ the sensitivity weighting matrix with respect to wind gust

M_1 = diag. (m_1 m_2 m_3 m_4) the sensitivity weighting matrix with respect to mass.

For ease of implementation and low cost, the controller will be assumed to be of the form

$$u = -G\underline{x} \tag{2.25}$$

where G is a constant matrix independent of the initial conditions. Thus the overall control system configuration will be of the same form as that of Optimal Design problem Fig. 2.2 except that the control gains G have different numerical values.

The low sensitivity optimization problem is now formulated as:

Find the constant gain matrix G in equation (2.25) which minimizes the performance index (2.24) subject to the system equation constraint (2.13).

CHAPTER 3

THEORY OF LOW SENSITIVITY STATE FEEDBACK DESIGN

Section 3.1 Introduction

The problem of low sensitivity design in optimal control systems has been investigated by several authors. ([H1],[L1]) In reference [L1], Lamont and Kahne introduced the sensitivity vector in the system performance index and compared the results obtained for different definitions of sensitivity. In the minimization of a performance index containing a sensitivity vector, the optimal feedback control will in general be a function of the sensitivity variables. Hendricks and D'Angelo [H1] studied the case in which the control structure is fixed, i.e., a feedback structure which is a function only of the state variables is assumed. However, the algorithms they developed depend on the initial conditions of the system. Furthermore, the model they used contained only one variable parameter.

As stated in Chapter 2, the interest here is in optimal state-feedback control which is independent of the initial conditions. Also, there are two variable parameters involved in the system equation (2.13). The objective of the work reported here was to develop an analytical theory for the low sensitivity optimization problem as stated in Section 2.4.3. The statement of the class of control systems studied in this chapter is in Section 3.2. The vehicle control problem formulated in Chapter 2 is a special case of the general problem formulated in this chapter. In Section 3.3.1 it is shown how to convert the class of control problem in Section 3.2 into the matrix differential equation form such that the "state" and "control" variables are matrices, instead of vectors. The necessary conditions for the low sensitivity feedback gains are derived in Section 3.3.2 by using the matrix minimum principle [A1].

Throughout Chapters 3, 4, and the Appendices, the following notations will be used:

(1) Small letters represent vectors unless otherwise noted.

(2) Capital letters represent matrices.

(3) I_n means the identity nxn matrix and O_n means the null nxn matrix.

(4) $A > O_n$ means that A is a positive definite matrix of dimension n and $A \geq O_n$ means that A is a positive semidefinite matrix of dimension n.

(5) A block diagonal matrix is a partitioned matrix with submatrices not along the main block diagonal being the null matrices. If A is a block diagonal matrix with A_1, A_2, and A_3 along the main block diagonal then A is expressed by

$\Lambda = \text{diag.}\ (\Lambda_1, \Lambda_2, \Lambda_3)$

Section 3.2 Statement of the Optimization Problem

Consider a completely controllable [L3] linear time-invariant system described by the vector differential equation

$$\dot{x}(t) = A(w,m)x(t) + Bu(t) \tag{3.1}$$

$$x(0) = x_o \text{ given}$$

where x is a n-vector, the state

u is a l-vector, the control, $l \leq n$

A is a nxn matrix with nominal value A_o

B is a nxl constant matrix

w and m are constant scale system parameters

x_o is the initial condition on x(t).

The system performance index is

$$J = \tfrac{1}{2}\int_0^\infty [x^T(t;\Lambda_o)Q_1x(t;\Lambda_o) + u^T(t;\Lambda_o)R_1u(t;\Lambda_o) + s_w^T(t;\Lambda_o)W_1s_w(t;\Lambda_o) + s_m^T(t;\Lambda_o)M_1s_m(t;\Lambda_o)]\ dt \tag{3.2}$$

where $x(t;\Lambda_o)$ is the nominal trajectory

$s_w(t;\Lambda_o)$ is the trajectory sensitivity vector with respect to w, n-vector

$s_m(t;\Lambda_o)$ is the trajectory sensitivity vector with respect to m, n-vector

$Q_1 > 0_n$, nxn state weighting matrix

$R_1 > 0_n$, lxl control weighting matrix

$W_1 \geq O_n$, nxn sensitivity weighting matrix with respect to w

$M_1 \geq O_n$, nxn sensitivity weighting matrix with respect to m.

The control law is assumed to be a linear function of the state variables

$$u = -Gx \tag{3.3}$$

where G is a constant lxn matrix.

The problem is to find the optimal feedback gain G in (3.3) such that the performance index (3.2) is minimized subject to the equations of constraint (3.1). Remark: If W_1 and M_1 are null matrices, then the problem becomes the standard optimal regulator problem and the assumption that the control be of the form given in (3.3) is unnecessary.

Section 3.3 The Solution of the Low Sensitivity Optimization Problem

Section 3.3.1 Analysis

Differentiating equation (3.1) with respect to w gives

$$\frac{\partial \dot{x}}{\partial w} = A \frac{\partial x}{\partial w} + \frac{\partial A}{\partial w} x + B \frac{\partial u}{\partial w} \tag{3.4}$$

Assuming

A1) x is sufficiently smooth both in t and w.

A2) $s_w(o) = 0$, and $s_m(0) = 0$, i.e., the parameter variations doe not affect the trajectory at the initial time.

Then equation (3.4) gives the sensitivity equation of the system

$$\dot{s}_w = As_w + \frac{\partial A}{\partial w} x + B \frac{\partial u}{\partial w} \tag{3.5}$$

$$s_w(0) = 0$$

Here the values of A, $\frac{\partial A}{\partial w}$, and $\frac{\partial u}{\partial w}$ are evaluated at the nominal values of w and m. Henceforth, it is understood that all matrices are evaluated at the nominal values of w and m.

From equation (3.3), it is found that

$$\frac{\partial u}{\partial w} = -Gs_w \tag{3.6}$$

Thus, the sensitivity equation with respect to w becomes

$$\dot{s}_w = A_w x + (A-BG)s_w \tag{3.7}$$

where $A_w = \frac{\partial A}{\partial w}$

In a similar manner, the sensitivity equation with respect to m is

$$\dot{s}_m = A_m x + (A-BG)s_m \tag{3.8}$$

$$s_m(0) = 0$$

Therefore, the compensated system and sensitivity equations are

$$\begin{cases} \dot{x} = (A-BG)x & x(0) = x_o \\ \dot{s}_w = A_w x + (A-BG)s_w & s_w(0) = 0 \\ \dot{s}_m = A_m x + (A-BG)s_m & s_m(0) = 0 \end{cases} \tag{3.9}$$

The performance index is

$$J = \tfrac{1}{2}\int_o^\infty [x^T(Q_1 + G^T R_1 G)x + s_w^T W_1 s_w + s_m^T M_1 s_m]\, dt \tag{3.10}$$

Define 3n-vector, y, as

$$y \triangleq \begin{bmatrix} x \\ s_w \\ s_m \end{bmatrix}$$

Then equations (3.9) and (3.10) become

$$\dot{y} = \hat{A}\, y \tag{3.11}$$

$$y^T(0) = (x_o \quad 0 \quad 0) \tag{3.12}$$

$$J = \tfrac{1}{2}\int_o^\infty y^T \text{diag.}(Q_1 + G^T R_1 G,\ W_1,\ M_1) y\, dt \tag{3.13}$$

where

$$\hat{A} = \begin{bmatrix} A-BG & 0 & 0 \\ A_w & A-BG & 0 \\ A_m & 0 & A-BG \end{bmatrix}$$

Let $\Phi(t,t_o)$ $(t_o = 0)$ be the fundamental matrix associated with equation (3.11). That is,

$$\dot{\Phi}(t,t_o) = \hat{A}\,\Phi(t,t_o) \tag{3.14}$$

$$\Phi(t_o, t_o) = I_{3n} \tag{3.15}$$

Since (A-BG), A_w and A_m are constant square matrices of the same dimension, we may partition $\Phi(t, t_o)$ into*

$$\Phi(t, t_o) = \begin{bmatrix} \Phi_1(t, t_o) & 0 & 0 \\ \Phi_2(t, t_o) & \Phi_1(t, t_o) & 0 \\ \Phi_3(t, t_o) & 0 & \Phi_1(t, t_o) \end{bmatrix} \tag{3.16}$$

where $\Phi_i(t, t_o)$ is nxn matrix.

Then equation (3.15) implies

$$\begin{aligned} \Phi_1(t_o, t_o) &= I_n \\ \Phi_2(t_o, t_o) &= \Phi_3(t_o, t_o) = 0_n \end{aligned} \tag{3.17}$$

From (3.12) and (3.16), the solution of equation (3.11) is

$$y(t) = \Phi(t, t_o) y_o = \begin{bmatrix} \Phi_1(t, t_o) \\ \Phi_2(t, t_o) \\ \Phi_3(t, t_o) \end{bmatrix} x_o \tag{3.18}$$

Substituting (3.18) into (3.13) and using formula (B.5) in Appendix B, we get

$$J = x_o^T \left(\tfrac{1}{2} \int_0^\infty [\Phi_1^T (Q_1 + G^T R_1 G) \Phi_1 + \Phi_2^T W_1 \Phi_2 + \Phi_3^T M_1 \Phi_3] dt \right) x_o$$

*Proof given in Appendix A.

$$= \frac{1}{2}\int_0^\infty \mathrm{tr.}([\Phi_1^T(Q_1 + G^T R_1 G)\Phi_1 + \Phi_2^T W_1 \Phi_2 +$$

$$\Phi_3^T M_1 \Phi_3][x_o x_o^T])\, dt \tag{3.19}$$

This expression shows the dependence of the performance index on both the constant gains G and the initial state of the system x_o. Since we are interested in feedback gains which are independent of x_o, it is necessary to eliminate this dependence on x_o.

To do this, consider a design which is in a sense "optimal" over some set of initial conditions. We will assume the initial condition vector in (3.19) to be a vector of random variable with

$$E(x_o) = 0 \tag{3.20}$$

and

$$E(x_o x_o^T) = X_o \tag{3.21}$$

The matrix X_o depends on a priori knowledge of the statistical distribution of the initial states. Since there is no such information available, the following choice is made. ([L2],[K4])

$$X_o = \lambda^2 I_n \qquad \lambda \text{ is constant} \tag{3.22}$$

This means that the vector of initial states x_o is uniformly distributed on the surface of the n-dimensional sphere of radius λ. Under this assumption the expected value of the performance index (3.19) is

$$\bar{J} = \frac{\lambda^2}{2} \int_0^\infty \text{tr.} \left[\Phi_1^T (Q_1 + G^T R_1 G) \Phi_1 + \Phi_2^T W_1 \Phi_2 + \Phi_3^T M_1 \Phi_3\right] dt \tag{3.23}$$

The justificiations for using the average performance index (3.23) are given below. Consider the initial state vectors on the surface of radius λ, then the optimal gain is the one which minimizes (3.23). Since λ is arbitrary but constant, the use of (3.23) will give the desired optimal gain which is independent of the initial conditions. Since the performance index can be multiplied by a constant without changing the value of the optimal gain matrix, we will use the following performance index for the optimization problem.

$$\hat{J} = \frac{1}{2} \int_0^\infty \text{tr.} \left[\Phi_1^T (Q_1 + G^T R_1 G) \Phi_1 + \Phi_2^T W_1 \Phi_2 + \Phi_3^T M_1 \Phi_3\right] dt \tag{3.24}$$

By means of this artifice, the optimization problem may be written as follows:

Find the gain matrix G such that it minimizes the performance index (3.24) subject to the system constraints (3.14) and (3.15).

Section 3.3.2 Necessary Conditions for Optimality

The solution of the above low sensitivity optimization problem is summarized in the following theorem.

<u>THEOREM 3.1</u>

Let G^* be a real constant 1xn matrix. Let

$$A^* = (A - BG^*) \tag{3.25}$$

and assume that A^* is a stability matrix. Then, in order for G^* to be optimal, it is necessary that

$$G^* = R_1^{-1}B^T(K_{11} + K_{22} + K_{33}) \tag{3.26}$$

where

(1) K_{ij} are nxn submatrices of

$$\hat{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \qquad \text{3n x 3n matrix}$$

and $\hat{K}$ satisfies the following matrix Lyapunov equation

$$\hat{K}\hat{A}^* + \hat{A}^{*T}\hat{K} + \hat{Q} = 0 \tag{3.27}$$

(2)

$$\hat{A}^* = \begin{bmatrix} A^* & 0 & 0 \\ A_w & A^* & 0 \\ A_m & 0 & A^* \end{bmatrix} \qquad \text{3n x 3n matrix} \tag{3.28}$$

(3)

$$\hat{Q} = \begin{bmatrix} Q_1 - W_1 - M_1 + G^T R_1 G & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & M_1 \end{bmatrix} \qquad \text{3n x 3n matrix} \tag{3.29}$$

Proof:

Rewrite equation (3.24) as

$$\hat{J} = \tfrac{1}{2}\int_0^\infty \mathrm{tr.}\left(\begin{bmatrix} \Phi_1^T & \Phi_2^T & \Phi_3^T \\ 0 & \Phi_1^T & 0 \\ 0 & 0 & \Phi_1^T \end{bmatrix} \begin{bmatrix} Q_1-W_1-M_1+G^TR_1G & 0 & 0 \\ 0 & W_1 & 0 \\ 0 & 0 & M_1 \end{bmatrix} \begin{bmatrix} \Phi_1 & 0 & 0 \\ \Phi_2 & \Phi_1 & 0 \\ \Phi_3 & 0 & \Phi_1 \end{bmatrix} \right) dt$$

$$= \tfrac{1}{2}\int_0^\infty \mathrm{tr.}(\Phi^T \hat{Q} \Phi)\, dt \qquad (3.30)$$

Let $P(t)$ be the 3nx3n costate matrix associated with $\Phi(t)$. The Hamiltonian function for this problem is

$$H = \tfrac{1}{2}\, \mathrm{tr.}(\Phi^T \hat{Q} \Phi) + \mathrm{tr.}(\hat{A} \Phi P^T) \qquad (3.31)$$

The canonical equations yield

$$\dot{\Phi} = \frac{\partial H}{\partial P} = \hat{A}\Phi \qquad (3.32)$$

$$\dot{P} = -\frac{\partial H}{\partial \Phi} = -\hat{Q}\Phi - \hat{A}^T P \qquad (3.33)$$

Since G is unconstrained, it is necessary that

$$0 = \left.\frac{\partial H}{\partial G}\right|_{G^*} = \tfrac{1}{2}\frac{\partial}{\partial G}\, \mathrm{tr.}(\Phi^T \hat{Q} \Phi) + \left.\frac{\partial}{\partial G}\, \mathrm{tr.}(\hat{A} \Phi P^T)\right|_{G^*} \qquad (3.34)$$

The solution of $P(t)$ is of the form ([K2])

$$P(t) = \hat{K}\, \Phi(t) \qquad (3.35)$$

where $\hat{K}$ is 3nx3n constant matrix.

Combining equations (3.35), (3.32) with (3.33) to obtain

$$(\hat{K}\hat{A} + \hat{A}^T\hat{K} + \hat{Q})\Phi = 0 \tag{3.36}$$

Since this is true for any $t \geq t_o$, so

$$\hat{K}\hat{A} + \hat{A}^T\hat{K} + \hat{Q} = 0 \tag{3.37}$$

Partitioning the matrices in (3.35) into submatrices

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} \Phi_1 & 0 & 0 \\ \Phi_2 & \Phi_1 & 0 \\ \Phi_3 & 0 & \Phi_1 \end{bmatrix} \tag{3.38}$$

where P_{ij}'s and K_{ij}'s are nxn matrices.

Expanding equation (3.38), the following relations are obtained.

$$\begin{aligned}
P_{11} &= K_{11}\Phi_1 + K_{12}\Phi_2 + K_{13}\Phi_3 \\
P_{12} &= K_{12}\Phi_1 \\
P_{13} &= K_{13}\Phi_1 \\
P_{21} &= K_{21}\Phi_1 + K_{22}\Phi_2 + K_{23}\Phi_3 \\
P_{22} &= K_{22}\Phi_1 \\
P_{23} &= K_{23}\Phi_1 \\
P_{31} &= K_{31}\Phi_1 + K_{32}\Phi_2 + K_{33}\Phi_3 \\
P_{32} &= K_{32}\Phi_1 \\
P_{33} &= K_{33}\Phi_1
\end{aligned} \tag{3.39}$$

Expanding equation (3.34) and using formula (B6) and (B7) in Appendix B to get

$$\tfrac{1}{2}\frac{\partial}{\partial G}\,\mathrm{tr.}[\Phi_1^T(Q_1+G^TR_1G)\Phi_1+\Phi_2^TW_1\Phi_2+\Phi_3^TM_1\Phi_3]$$
$$+\frac{\partial}{\partial G}\,\mathrm{tr.}[(A-BG)\Phi_1P_{11}^T]+\frac{\partial}{\partial G}\,\mathrm{tr.}[A_w\Phi_1P_{21}^T+(A-BG)\Phi_2P_{21}^T]$$
$$+\frac{\partial}{\partial G}\,\mathrm{tr.}[(A-BG)\Phi_1P_{22}^T]+\frac{\partial}{\partial G}\,\mathrm{tr.}[A_m\Phi_1P_{31}^T+(A-BG)\Phi_3P_{31}^T]$$
$$+\frac{\partial}{\partial G}\,\mathrm{tr.}[(A-BG)\Phi_1P_{33}^T]\Big|_{G=G^*}=0 \tag{3.40}$$

Using formula (B11) and (B14), equation (3.40) is reduced to

$$R_1G^*\Phi_1\Phi_1^T-B^TP_{11}\Phi_1^T-B^TP_{21}\Phi_2^T-B^TP_{22}\Phi_1^T$$
$$-B^TP_{31}\Phi_3^T-B^TP_{33}\Phi_1^T\Big|_{G=G^*}=0$$

or

$$G^*=R_1^{-1}B^T(P_{11}\Phi_1^T+P_{21}\Phi_2^T+P_{31}\Phi_3^T+P_{33}\Phi_1^T)(\Phi_1\Phi_1^T)^{-1} \tag{3.41}$$

Substituting equation (3.39) into equation (3.41) gives

$$G^*=R_1^{-1}B^T[K_{11}+K_{22}+K_{33}+(K_{12}\Phi_2+K_{13}\Phi_3)\Phi_1^{-1}$$
$$+(K_{21}\Phi_1+K_{22}\Phi_2+K_{23}\Phi_3)(\Phi_2^T\Phi_1^{T^{-1}}\Phi_1^{-1})$$
$$+(K_{31}\Phi_1+K_{32}\Phi_2+K_{33}\Phi_3)(\Phi_3^T\Phi_1^{T^{-1}}\Phi_1^{-1})] \tag{3.42}$$

Since G^* is constant, let $t = t_o$ in equation (3.42) and use equation (3.17), then (3.42) becomes

$$G^*=R_1^{-1}B^T(K_{11}+K_{22}+K_{33}) \tag{3.43}$$

where

$$\hat{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

satisfies the matrix Lyapunov equation (3.37).

This completes the proof.

<u>Remark</u>:

If $W_1 = M_1 = O_n$, then $K_{ij} = 0$ except K_{11}. Equations (3.26) and (3.27) become

$$G^* = R_1^{-1}B^TK_{11}$$

$$K_{11}A_o + A_o^TK_{11} - K_{11}BR_1^{-1}B^TK_{11} + Q_1 = 0$$

This is the Riccati equation in the optimal design case. (pages 2-11)

Thus, the optimal design may be viewed as a special case of the low sensitivity design where W_1 and M_1 are null matrices.

CHAPTER 4

THEORY OF LOW SENSITIVITY STOCHASTIC STATE FEEDBACK DESIGN

Section 4.1 Introduction

The problems discussed so far are confined to deterministic cases in which the exact measurement of all state variables is necessary. In actual practice, it is inconvenient and expensive to measure accurately the acceleration error and the rate of change of propulsive force. In reference [G4] Garrard and Kornhauser introduced the theory of state observers to estimate all of the state variables from measurements of only position and velocity errors. However, perfect sensing was assumed in their development.

In this chapter, estimation and control theories are combined and applied to the design of a longitudinal control system for automated transit vehicles. The design process includes not only imperfect sensing but also the disturbance environment. ([A4]) In Section 4.2, the longitudinal control problem in the presence of external disturbances and sensor errors is formulated. In Section 4.2.2, the optimal design problem based on average values of parameters is discussed. The low sensitivity design problem is presented in Section 4.2.3. Finally, the solution of the low sensitivity design problem is discussed in Section 4.3.

Section 4.2 The Longitudinal Control Problem Formulation

Section 4.2.1 Assumptions

The following assumptions are made in this study:

(A1) The sensors contain no dynamics. This means that the measured quantities are the actual quantities with uncertainty due to imperfect sensing.

(A2) The position and velocity measurements are made frequently enough such that the discrete measurement process can be viewed as the continuous measurement process.

(A3) The sensor errors and external disturbances which are not generated by the control system are modeled as additive random processes.

(A4) The external disturbance $d(t)$ is a white Gaussian process with zero mean and constant covariance matrix Q_2. This means that $d(t)$ is Gaussian for each t. Therefore its statistical properties are uniquely determined by its mean and variance. The assumption of a white noise process implies that the disturbance $d(t)$ is uncorrelated for distinct times. Since the process is Gaussian, the values of $d(t)$ at distinct times are independent. Thus the past knowledge of the input disturbance $d(t)$ does not help in predicting the future values of $d(t)$. In mathematical notation, the input disturbance $d(t)$ is described by

$$E[d(t)] = 0 \tag{4.1}$$

$$\mathrm{cov}[d(t);d(\tau)] = Q_2\ \delta(t-\tau) \tag{4.2}$$

where $Q_2 = Q_2^T > 0$ and δ is the Dirac delta function defined by ([B2])

$$\delta(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t) \tag{4.3}$$

$$\delta_\epsilon(t) = \begin{cases} 0 \quad , & |t| > \epsilon \\ \frac{1}{2\epsilon} \quad , & |t| < \epsilon \end{cases}$$

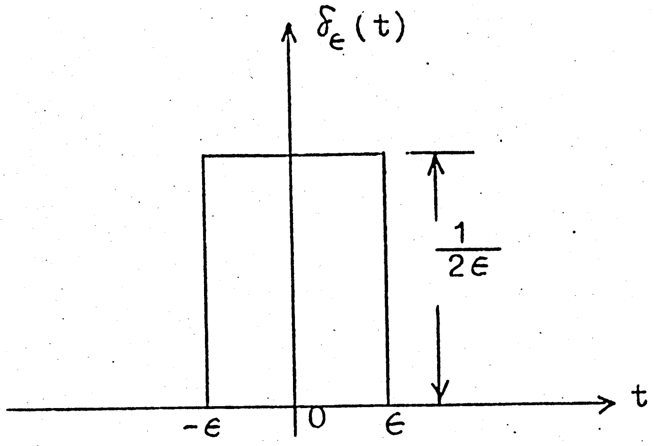


Fig. 4.1 Definition of $\delta_\epsilon(t)$

(A5) The sensor error $v(t)$ is a white Gaussian process with zero mean and constant covariance matrix R_2. Physically this means that the sensor measurement error $v(t_1)$ is independent of the past measurement error $v(t_2)$, $t_2 < t_1$. That is,

$$E[v(t)] = 0 \tag{4.4}$$

$$\mathrm{cov}[v(t);v(\tau)] = R_2\ \delta(t-\tau) \tag{4.5}$$

where $R_2 = R_2^T > 0$.

(A6) The initial state vector $x(t_o)$ is Gaussian with known mean and constant covariance matrix.

$$E[x(t_o)] = \bar{x}_o \tag{4.6}$$

$$\mathrm{cov}[x(t_o);x(t_o)] = X_o \tag{4.7}$$

(A7) The process $v(t)$, $d(t)$, and $x(t_o)$ are mutually independent. This implies that these processes are uncorrelated to each other, i.e.,

$$\begin{aligned} &\mathrm{cov}[d(t);v(\tau)] = 0 \\ &\mathrm{cov}[d(t);x(t_o)] = 0 \\ &\mathrm{cov}[v(t);x(t_o)] = 0 \quad \text{for all } t,\ \tau \geq t_o \end{aligned} \tag{4.8}$$

Under these assumptions, the longitudinal control system equations can be written as

$$\dot{x}(t) = Ax(t) + Bu(t) + d(t) \tag{4.9}$$

$$z(t) = Cx(t) + v(t) \tag{4.10}$$

where $z(t)$ is the output from the sensors and

$C = [1 \quad 0 \quad 0 \quad 0]$ if only position is measured,

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ if both position and velocity are measured.

Note that the system described by (4.9)-(4.10) is completely controllable as discussed in Section 2.4.2.

It is completely observable ([L3]) if the position measurement is made, since

$$\text{rank}[C^T, A^TC^T, (A^T)^2C^T, (A^T)^3C^T] = 4 \qquad (4.11)$$

Section 4.2.2 The Optimal Design Problem

As in the previous chapter the design based on average values of the vehicle parameters will be called the Optimal Design. The proposed performance index for the optimal design in the presence of external disturbances and sensor errors is

$$J_1 = E\left[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (x^TQ_1x + u^TR_1u)dt\right] \qquad (4.12)$$

The rationale for choosing the performance index in the quadratic form is the same as discussed in Section 2.4.2. Since $x(t)$ and $u(t)$ are random variables, the criterion is taken to be the expected value of the weighted sums. Due to the presence of sensor errors and input disturbances, the state variables, which represent the system error, will in general be non-zero. The integral $\frac{1}{2}\int_0^\infty (x^TQ_1x + u^TR_1u)dt$ will become infinite for nonzero $x(t)$. Thus, the performance index will be chosen to be the average value of the integral (2.16), i.e.,

$$\lim_{t_f \to \infty} \left(\frac{1}{t_f}\right)\left(\frac{1}{2}\int_0^{t_f} (x^TQ_1x + u^TR_1u)dt\right) \qquad (4.13)$$

The design goal is to keep the state variables $x(t)$ close to zero by proper selection of the control $u(t)$. Since all the state variables cannot be measured, an estimator is needed for generating an estimate of the state, denoted as $\hat{x}(t)$, from the available signal $z(\tau)$ for all $\tau \in [t_o, t]$. This estimator should be best in a well-defined statistical sense. In our design problem, the optimal estimator will be studied in the sense of minimizing the mean square error of estimation

$$J_2 = E[(x-\hat{x})^T(x-\hat{x})] \tag{4.14}$$

The combined estimation and control problem is to find the control u(t) and the best estimator such that J_1 is minimized subject to the system equations (4.9)-(4.10).

This is the standard Linear-Quadratic-Gaussian (LQG) problem of stochastic control theory. The separation theorem, which says that the overall LQG problem solution can be separated into the solution of a linear-quadratic deterministic problem and the solution of a linear-Gaussian estimation problem, is applicable. ([A3],[B2], [S1], [W4]). The optimal estimator is the Kalman-Bucy filter and the optimal control is

$$u = -G\hat{x} \tag{4.15}$$

where (1) G is the optimal gain matrix in the deterministic case, i.e.,

$$G = R_1^{-1}B^TK_1 \tag{4.16}$$

K_1 satisfies the matrix Riccati equation

$$K_1A_o + A_o^TK_1 - K_1BR_1^{-1}B^TK_1 + Q_1 = 0 \tag{4.17}$$

(2) $\hat{x}(t)$, the unbiased estimate of the state $x(t)$, is the output of the estimator

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H_2[z(t) - C\hat{x}(t)]$$
$$= (A-BG-H_2C)\hat{x}(t) + H_2z(t) \qquad \hat{x}(t_o) = \bar{x}_o \tag{4.18}$$

where the filtering matrix H_2 is given by

$$H_2 = K_2C^TR_2^{-1} \tag{4.19}$$

K_2 is the covariance matrix of estimation error, satisfying the matrix Riccati equation

$$K_2A_o^T + A_oK_2 - K_2C^TR_2^{-1}CK_2 + Q_2 = 0 \tag{4.20}$$

The covariance matrix of the estimated state $\hat{X}$ is given by

$$(A-BG)\hat{X} + \hat{X}(A-BG)^T + H_2R_2H_2^T = 0 \tag{4.21}$$

The covariance matrix of the state X is

$$X = \hat{X} + K_2 \tag{4.22}$$

The covariance matrix of the control variable U is given by

$$U = G\hat{X}G^T \tag{4.23}$$

The overall control diagram is as shown in Fig. 4.2.

Section 4.2.3 The Low Sensitivity Design Problem

The design which includes the sensitivity of vehicle dynamic response due to changes in system parameters will

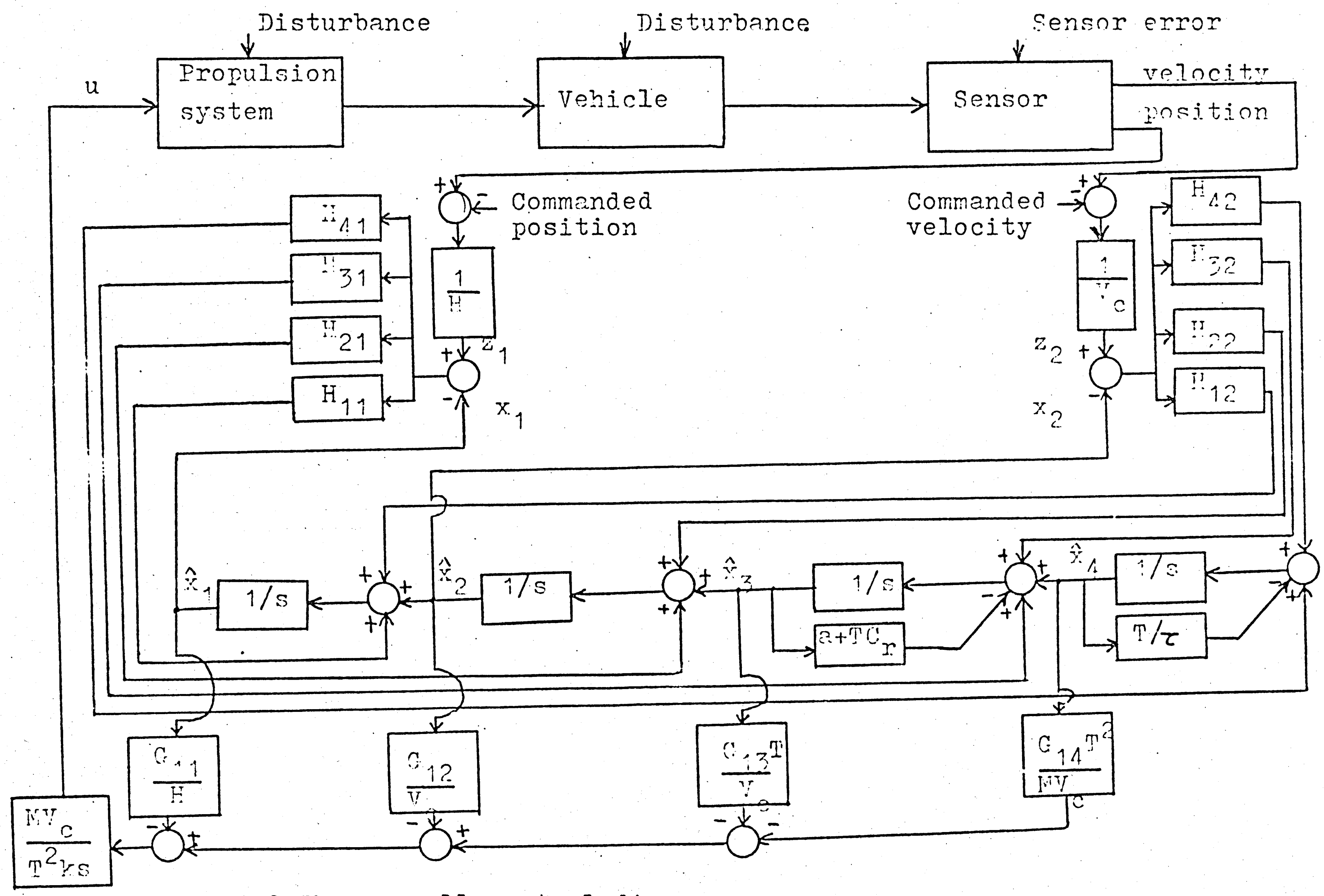


Fig. 4.2 The overall control diagram

be investigated in this section. In order for the results presented below to be correct in a rigorous mathematical sense, the following assumption must be satisfied.

(A8) The parameters w, m, and the disturbances d(t), v(t) must be mutually independent.

Of course the input disturbance d(t) and the wind gust parameter w will not be independent for disturbances which are caused by wind gusts; however, it is shown by means of computer simulations that even though assumption (A8) is violated in certain cases, the controllers resulting from the application of the theory developed below yield excellent dynamic response in a variety of realistic situations.

As in Chapter 3 the low sensitivity design is accomplished by modifying the performance index to include a measure of the trajectory sensitivity vectors

$$J_1 = E\left[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (x^T Q_1 x + u^T R_1 u + s_w^T W_1 s_w + s_m^T M_1 s_m)\, dt \right] \tag{4.24}$$

where s_w and s_m are the trajectory sensitivity vectors, defined by

$$s_w \triangleq \left. \frac{\partial \hat{x}}{\partial w} \right|_{(w,m)=(0,1)} \tag{4.25}$$

$$s_m \triangleq \left. \frac{\partial \hat{x}}{\partial m} \right|_{(w,m)=(0,1)} \tag{4.26}$$

Here $\hat{x}$ is the estimate of state. Note that the definition of sensitivity vectors is taken with respect to the estimated state $\hat{x}(t)$ which is called the accessible state in the literature. ([T1]) The actual state $x(t)$ is called the inaccessible state because it cannot be measured from sensors.

The feedback control structure of interest is assumed to be of a form which is motivated by the standard LQG problem solution in the previous section. That is,

$$u = -G\hat{x} \tag{4.27}$$

where G is the constant controller

$\hat{x}$ is the output of the optimal estimator in the sense of minimizing

$$J_2 = E[(x-\hat{x})^T(x-\hat{x})] \tag{4.28}$$

The low sensitivity stochastic design problem is to find the control gain G and the optimal estimator such that (4.24) and (4.28) are minimized.

Before proceeding to the next section, let us note that the linear-Gaussian assumptions imply the processes $x(t)$ and $z(t)$ are also Gaussian. Therefore, the estimate $\hat{x}(t)$ which minimizes (4.28) is the same as the conditional mean given the past accumulative observation. ([K6, p. 37]) In mathematical notation, it is

$$\hat{x}(t) = E[x(t) \,|\, \mathcal{F}_t] \tag{4.29}$$

where

$$\mathcal{F}_t = \left\{ z(\tau): \tau \in [0, t] \right\}$$

Section 4.3 The Solution of the Low Sensitivity Stochastic Design Problem

The derivation of the low sensitivity stochastic design problem stated in the previous section will be divided into three steps for clarity.

Section 4.3.1 Derivation of the Optimal Estimator

The linear-Gaussian assumption plays an important role in the derivation of the optimal estimator. The definition of the admissible controls is necessary in this step.

Definition: The control u(t) is called admissible if it can only depend on the past accumulative observation data, i.e.,

$$u(t) = \psi \left[t, z(\tau), \tau \in [0, t] \right] \qquad (4.30)$$

The following two Lemmas will be used in the development of the optimal estimator for the low sensitivity stochastic design problem. The proof of the Lemmas may be found in the references cited below and are not given here.

<u>Lemma 1</u> ([W4, p. 314])

Let $\hat{\psi}(t,\hat{x}(t))$ be the class of functions

$$\hat{\psi}: [0,T] \times R^n \longrightarrow U$$

such that

$$\left|\hat{\psi}(t_1,\hat{x}_1) - \hat{\psi}(t_2,\hat{x}_1)\right| + \left|\hat{\psi}(t_1,\hat{x}_1) - \hat{\psi}(t_1,\hat{x}_2)\right| \le$$

$$c_2(R)\left|t_1-t_2\right|^{\alpha} + c_3\left|\hat{x}_1-\hat{x}_2\right| \qquad (4.31)$$

in every domain $0 \le t_1, t_2 \le T$, $|\hat{x}_1| < R$, $|\hat{x}_2| < R$ where c_3 and $\alpha \in (0,\frac{1}{2})$ are independent of R. Then, $u(t) = \hat{\psi}(t,\hat{x}(t))$ is admissible.

<u>Lemma 2</u> ([T1, p. 778])

Consider the system described by (4.10) is controllable and observable and assumptions (A3)-(A8) hold. If u(t) is admissible, then the corresponding conditional distribution of the state x(t) is Gaussian with conditional mean $\hat{x}(t)$ and conditional covariance matrix K_2 given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H_2[z(t) - C\hat{x}(t)] \quad \hat{x}(t_o) = \bar{x}_o \qquad (4.32)$$

$$H_2 = K_2C^TR_2^{-1} \qquad (4.33)$$

$$AK_2 + K_2A^T + Q_2 - K_2C^TH_2^{-1}CK_2 = 0 \qquad (4.34)$$

The control structure of the low sensitivity design is assumed to be in the form (4.27). In the following it will be proved that the control structure (4.27) is admissible.

Define the norm of matrix G to be ([R3])

$$\|G\| = \underset{x \leq 1}{\text{lub}} \; |Gx|$$

Then the following inequality holds

$$\left|G\hat{x}_1 - G\hat{x}_2\right| \leq \|G\| \cdot \left|\hat{x}_1 - \hat{x}_2\right| \tag{4.35}$$

By choosing in Lemma 1

$$c_3 = \|G\|$$

$$\alpha = \text{any value in } (0, \tfrac{1}{2})$$

Thus, $u = -G\hat{x}$ is admissible as seen from Lemma 1. Since (4.27) is admissible, the optimal estimate $\hat{x}(t)$ is obtained by applying Lemma 2, i.e.,

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + H_2[z(t) - C\hat{x}(t)] \quad \hat{x}(t_o) = \bar{x}_o \tag{4.36}$$

$$H_2 = K_2 C^T R_2^{-1} \tag{4.37}$$

$$AK_2 + K_2 A^T - K_2 C^T R_2^{-1} C K_2 + Q_2 = 0 \tag{4.38}$$

where

$$K_2 = E[(x-\hat{x})(x-\hat{x})^T] \tag{4.39}$$

= covariance matrix of the estimation error

Section 4.3.2 Derivation of the Low Sensitivity Controller

Define the estimation error variable

$$\tilde{x}(t) = x(t) - \hat{x}(t) \tag{4.40}$$

Since $\hat{x}(t)$ is the best estimate of $x(t)$ in the sense of minimizing mean square error as in (4.28), it is orthogonal to the estimation error such that the covariance of the estimate and the estimation error is zero. ([K5, p. 103], [S1, p. 227]). That is,

$$\mathrm{cov}[\tilde{x}(t);\hat{x}(t)] = 0 \tag{4.41}$$

Equation (4.41) is called the orthogonal projection lemma in the literature.

By subtracting equation (4.36) from equation (4.9), the equation governing the estimation error is obtained as

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) - H_2[C\tilde{x}(t) + v(t)] + d(t) \tag{4.42}$$

Define the process $\eta(t)$ to be

$$\eta(t) = z - C\hat{x} = C\tilde{x} + v \tag{4.43}$$

$$= \text{correction terms of the optimal estimator}$$

The process $\eta(t)$ is called the innovation process. ([K7], [R2], [T1]). It can be shown that innovation process $\eta(t)$ is a white Gaussian process with zero mean and the same covariance matrix as the process $v(t)$ under linear-Gaussian assumptions. Rigorous proofs of this statement can be found in numerous references. ([K7], [B2], [R2], [M3]). In terms of statistical notation, the mean and covariance of the process $\eta(t)$ are

$$E[\eta(t)] = 0 \tag{4.44}$$

$$\mathrm{cov}[\eta(t);\eta(\tau)] = R_2\,\delta(t-\tau) \tag{4.45}$$

The differential equations for $\tilde{x}(t)$ and $\hat{x}(t)$ in terms of the innovation process $\eta(t)$ are

$$\dot{\hat{x}} = A\hat{x} + Bu + H_2\eta \tag{4.46}$$

$$\dot{\tilde{x}} = A\tilde{x} - H_2\eta + d \tag{4.47}$$

Thus, the estimated error $\tilde{x}(t)$ is unaffected by the control $u(t)$ as seen from (4.47). Using (4.41) the performance index (4.24) may be written as

$$J_1 = E[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (\hat{x}^T Q_1 \hat{x} + u^T R_1 u + s_w^T W_1 s_w + s_m^T M_1 s_m) dt]$$

$$+ E[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (\tilde{x}^T Q_1 \tilde{x})\, dt] \tag{4.48}$$

The expression (4.48) may be written in terms of the covariance matrix of estimation error K_2 by substituting in formula (B5) in Appendix B.

$$J_1 = E[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (\hat{x}^T Q_1 \hat{x} + u^T R_1 u + s_w^T W_1 s_w + s_m^T M_1 s_m) dt]$$

$$+ \tfrac{1}{2} Q_1 K_2 \tag{4.49}$$

Thus the performance index of the original stochastic control problem with inaccessible states $x(t)$ consists of two parts as shown in (4.49). Since K_2 is the solution of the algebraic Riccati matrix equation (4.38), it is deterministic and independent of the observation $z(t)$. The second part in (4.49) is, therefore, independent of the control. This and the fact that the estimation error $\tilde{x}(t)$ is unaffected by the control variables imply that the problem of minimizing J_1 subject to (4.9)-(4.10) is equivalent to that of minimizing the first part of (4.49) subject to equation (4.46). This equivalent stochastic control problem with accessible state $\hat{x}(t)$ is written below for clearness.

System eq. $$\dot{\hat{x}} = A\hat{x} + Bu + H_2 \eta \tag{4.50}$$

P.I. $$J_3 = E\left[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (\hat{x}^T Q_1 \hat{x} + u^T R_1 u + s_w^T W_1 s_w + s_m^T M_1 s_m) dt \right] \tag{4.51}$$

Control law $$u = -G\hat{x} \tag{4.52}$$

The control problem is to find the control gain in (4.52) such that (4.51) is minimized subject to the system equation constraint (4.50).
Since the innovation process $\eta(t)$ represents the random input Gaussian disturbance with zero mean and very short correlation times (white noise), it is impossible to predict $\eta(\tau)$ for $\tau > t$ even with perfect knowledge of the accessible states for $\tau < t$. Therefore, the stochastic control problem with accessible states (4.50)-(4.52) is equivalent to the deterministic problem of finding the control gain G. ([B2]). That is,

find $$u = -G\hat{x} \tag{4.53}$$

such that it minimizes

$$J_3 = E[\lim_{t_f \to \infty} \frac{1}{2t_f} \int_0^{t_f} (\hat{x}^T Q_1 \hat{x} + u^T R_1 u + s_w^T W_1 s_w + s_m^T M_1 s_m) dt] \tag{4.54}$$

subject to

$$\dot{\hat{x}} = A\hat{x} + Bu \tag{4.55}$$

This is the deterministic low sensitivity design problem as studied in the previous chapter. The resulting necessary conditions for optimal gain G^* are given in Theorem 3.1 (page 3.10). That is,

$$G^* = R_1^{-1} B^T (K_{11} + K_{22} + K_{33}) \tag{4.56}$$

where

(1) K_{ij}'s are submatrices of

$$\hat{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$$

$\hat{K}$ is 3nx3n matrix satisfying the Lyapunov equation

$$\hat{K}\hat{A}^* + \hat{A}^{*T}\hat{K} + \hat{Q} = 0 \tag{4.57}$$

(2)

$$\hat{A}^* = \begin{bmatrix} A^* & 0 & 0 \\ A_w & A^* & 0 \\ A_m & 0 & A^* \end{bmatrix} \quad (w,m) = (0,1)$$

and

$$A^* = (A - BG^*)$$

(3)

$$\hat{Q} = \text{diag.}(Q_1 - W_1 - M_1 + G^T R_1 G,\ W_1,\ M_1)$$

Section 4.3.3 Average Behavior of the Control System

It is desired to derive the average behavior of the control system for the low sensitivity stochastic design problem. The covariance matrix of the estimation error is as given by (4.39)

$$E[\tilde{x}(t)\tilde{x}^T(t)] = K_2 \tag{4.58}$$

where K_2 satisfies the Riccati equation (4.38). Using equations (4.36), (4.45), and (4.50) the covariance matrix $\hat{X}$ for the estimated state $\hat{x}(t)$ may be obtained from

$$(A-BG)\hat{X} + \hat{X}(A-BG)^T + H_2 R_2 H_2^T = 0 \tag{4.59}$$

The covariance matrix for the state X is obtained by applying the orthogonal projection lemma (4.41), i.e.,

$$\begin{aligned} X &= \text{cov}[x(t);x(t)] \\ &= \text{cov}[\hat{x}(t);\hat{x}(t)] + \text{cov}[\tilde{x}(t);\tilde{x}(t)] \\ &= \hat{X} + K_2 \end{aligned} \tag{4.60}$$

From (4.53) the covariance matrix of the control variable is

$$U = G\hat{X}G^T \tag{4.61}$$

Section 4.3.4 Summary of the Solution of the Low Sensitivity Stochastic Design Problem

The solution of the stochastic low sensitivity design problem stated in Section 4.2.3 (page 4.10) is

$$u = -G^{*}\hat{x} \tag{4.62}$$

where

(1) G^{*} is the deterministic low sensitivity design gain, i.e.,

$$G^{*} = R_1^{-1}B^{T}(K_{11} + K_{22} + K_{33}) \tag{4.63}$$

and $\hat{K} = [K_{ij}]_{\substack{i=1,3\\ j=1,3}}$ satisfies

$$\hat{K}\hat{A}^{*} + \hat{A}^{*T}\hat{K} + \hat{Q} = 0 \tag{4.64}$$

(2) $\hat{x}(t)$ is the output of the optimal estimator, which is the Kalman-Bucy filter, i.e.,

$$\dot{\hat{x}} = A\hat{x} + Bu + H_2(z-C\hat{x}) \qquad \hat{x}(t_o) = \bar{x}_o \tag{4.65}$$

where the filtering matrix H_2 is given by

$$H_2 = K_2C^{T}R_2^{-1} \tag{4.66}$$

K_2 is the covariance matrix of the estimation error, satisfying the matrix Riccati equation

$$K_2A^{T} + AK_2 - K_2C^{T}R_2^{-1}CK_2 + Q_2 = 0 \tag{4.67}$$

(3) The covariance matrix of the estimated state $\hat{X}$ is

$$(A-BG^{*})\hat{X} + \hat{X}(A-BG^{*})^{T} + H_2R_2H_2^{T} = 0 \tag{4.68}$$

The covariance matrix of the state is

$$X = \hat{X} + K_2 \tag{4.69}$$

The covariance matrix of the control is

$$U = G^{*}\hat{X}G^{*T} \tag{4.70}$$

The overall control diagram is as shown in Fig. 4.2 except different numerical values are used for the control gains.

<u>Remark</u>: The covariance matrix of the estimation error K_2 and the filtering matrix H_2 are the same for both the optimal design and the low sensitivity design. However, the other matrices G, $\hat{X}$, X, and U are different.

CHAPTER 5

EXAMPLE: DESIGN OF AN OPTIMAL LONGITUDINAL FEEDBACK CONTROL SYSTEM FOR A HIGH-CAPACITY PRT SYSTEM

Section 5.1 Introduction

An example, using the theory developed in previous chapters, is studied in this chapter. The problem is to design an optimal longitudinal control system for a high-capacity PRT system. The vehicle data and control system specifications are given in Section 5.2. Section 5.3 shows how to select the gains for the control system. Simulation results for normal mainline operations, and acceleration profile-following operations are presented in Section 5.4. The simulations in the stochastic cases were studied in Section 5.5. In Section 5.5.1 the selection of covariance matrices Q_2 and R_2 is discussed. The effect of eliminating the velocity sensor is studied in Section 5.5.2. The effects of input disturbances and sensor errors upon the control system performance for various designs are investigated in Section 5.5.3. Finally, the comparisons between the control system using estimators (Kalman-Bucy filters) and the control system using observers are made in Section 5.5.4.

Section 5.2 Vehicle Data and System Specifications

A longitudinal control system was designed for a high-capacity PRT system with the following typical vehicle data and control system specifications: ([A7], [G1], [V1], [S3])

Vehicle length = 10 ft

Drag coefficient $C_a = .025\ \text{lbf}\cdot\text{sec}^2/\text{ft}^2$

Rolling resistance coefficients $C_s = .245\ \text{ft/sec}^2$

$C_r = .000197/\text{sec}$

Average vehicle mass including passengers and goods = 4830 lbs

2415 lbs $\leq$ vehicle mass $\leq$ 7245 lbs

Maximum wind gust velocity = $\pm$ 150 ft/sec

Normal mainline velocity = 50 ft/sec

Minimum headway time = 1 sec

Propulsion system time constant = .1 sec

Maximum acceleration for normal operation = $\pm$ .125g

Maximum acceleration for merging and maneuvering operation = $\pm$ .25g

Maximum deceleration for emergency stopping operation = -.35g

Maximum jerk for normal operation = $\pm$ .1g/sec

Maximum jerk for merging and maneuvering operation = $\pm$ .25g/sec

Maximum jerk for emergency stopping operation = -.35g/sec

Control system velocity tolerance = $\pm$ 2.5 ft/sec

Control system position tolerance = $\pm$ 5 ft

The above specifications correspond to a system with a maximum theoretical capacity of 3600 vehicles per hour.

Section 5.3 Selection of Control Gains

Section 5.3.1 Determination of Weighting Matrices Q_1, W_1, and M_1

The weighting factors (the q's, w's, m's, and r) affect the values of the elements of the gain matrix which in turn affect the dynamic response of the vehicle. The relationship between the weighting factors and the dynamic response of the vehicle can not be analytically determined. Preliminary computational results indicated that the velocity error should be weighted ten times the position error ($q_2 = 10q_1$). This resulted in tighter headway control as well as a more comfortable ride than could be obtained by weighting the position error the same as or greater than the velocity error. ([G1]). Since the performance index can be multiplied by a constant without changing the value of the gain matrix, the weighting factor on the control was chosen to be one. The weighting factor on the acceleration error, q_3, was set equal to ten and the value of the position error weighting

factor was varied with respect to q_3. Since the absolute value of the maximum jerk has a numerical value nearly equal to the absolute value of the maximum acceleration, the weighting factor associated with the jerk, q_4, was also set equal to ten.
Since the variation of the vehicle response with respect to the variable parameters w and m is much smaller than the absolute value of the state variables (see Figures (5.9)-(5.12)), the sensitivity vectors must be weighted heavily in order to efficiently reduce the variation in dynamic response. The weighting matrices W_1 and M_1 are chosen to be equal to Q_1 and ten times Q_1.
The optimal gain of the longitudinal control system is (page 2.11)

$$G = \underline{b}^T K \tag{5.1}$$

where K is the solution of

$$KA_o + A_o^T K - K\underline{b}\ \underline{b}^T K + Q_1 = 0 \tag{5.2}$$

and

$$A_o = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -.0169 & 1 \\ 0 & 0 & 0 & -10 \end{bmatrix}$$

$$\underline{b}^T = (0 \quad 0 \quad 0 \quad 1)$$

$$Q_1 = \text{diag.}(q_1\ ,\ \ 10q_1\ ,\ \ 10,\ \ 10)$$

The low sensitivity design gain is (page 3.10)

$$G^* = \underline{b}^T(K_{11} + K_{22} + K_{33}) \tag{5.3}$$

where $\hat{K} = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}$ satisfies the equation

$$\hat{K}\hat{A}^* + \hat{A}^{*T}\hat{K} + \hat{Q} = 0 \tag{5.4}$$

and

$$\hat{A}^* = \begin{bmatrix} A_o-\underline{b}G^* & 0 & 0 \\ A_w & A_o-\underline{b}G^* & 0 \\ A_m & 0 & A_o-\underline{b}G^* \end{bmatrix}$$

$$A_w = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -.0167 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_m = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & .0167 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\hat{Q} = \text{diag.}\ (Q_1-W_1-M_1+G^{*T}G^*,\ W_1\ ,\ M_1)$$

An iterative scheme ([K3]) for the solution of the control gain was written for the CDC 6600 digital computer. The Flow Charts are given in Appendix C. The

q_1/q_3	G_{11}	G_{12}	G_{13}	G_{14}	
10^{-2}	.316	2.468	7.863	1.213	$W_1=M_1=0$
		2.552	8.527	1.304	$W_1=M_1=Q_1$
		2.899	11.496	1.862	$W_1=M_1=10Q_1$
10^{-1}	1.000	5.822	11.753	1.555	0
		5.996	12.779	1.668	Q_1
		6.706	17.272	2.331	$10Q_1$
10^{0}	3.162	14.864	18.917	2.159	0
		15.224	20.624	2.325	Q_1
		16.708	28.100	3.227	$10Q_1$
10^{1}	10.00	40.660	32.438	3.224	0
		41.412	35.519	3.501	Q_1
		44.497	48.742	4.840	$10Q_1$
10^{2}	31.62	117.19	58.794	5.086	0
		118.81	64.810	5.574	Q_1
		125.26	89.660	7.640	$10Q_1$
10^{3}	100.0	349.95	111.99	8.276	0
		353.51	124.54	9.143	Q_1
		367.14	173.56	12.34	$10Q_1$
10^{4}	316.2	1068.2	222.67	13.57	0
		1076.2	249.63	15.05	Q_1
		1105.1	349.32	19.89	$10Q_1$
10^{5}	1000.	3303.9	457.32	22.01	0
		3321.7	515.78	24.42	Q_1
		3383.9	727.73	31.74	$10Q_1$

Table I Nondimensional optimal and low sensitivity design gains for various q_1/q_3 ; $Q_1=\text{diag.}(q_1\ ,10q_1\ ,10,10)$

nondimensional gains for various values of q_1/q_3 are shown in Table I.

Section 5.3.2 Selection of the Control Gains

The minimum headway for the system is 50 feet, thus the position error tolerance is ten percent. Fig.(5.1) shows the nondimensional time required to reduce ten percent of initial position error of 5 feet as a function of the ratio q_1 to q_3. The maximum velocity error, maximum acceleration, and maximum jerk in reducing the position error of 5 feet to its normal state were plotted versus q_1/q_3 as shown in Figs.(5.2) -(5.4). In Fig.(5.2) it is seen that the velocity error is within the tolerance value for any q_1/q_3 chosen from 10^{-2} to 10^5. However, in order to have acceleration less than the maximum acceleration for normal operation ($\pm$ .125g), the ratio of q_1/q_3 must be less than 10^3 as indicated in Fig.(5.3). In the same manner, the maximum jerk for normal operation requires $q_1/q_3 \leq 10^1$.

The normal mainline velocity is 50 ft/sec, thus the velocity error tolerance is five percent. Figs. (5.5)-(5.8) are plotted for an initial velocity error of five percent. Fig.(5.6) shows that in order to obtain position error of less than the position tolerance, it is necessary that q_1/q_3 be at least 10^0. The maximum acceleration requirement implies $q_1/q_3 \leq 10^2$ as calculated from Fig.(5.7). From Fig. (5.8), q_1/q_3 should be less than or equal to 10^0 for

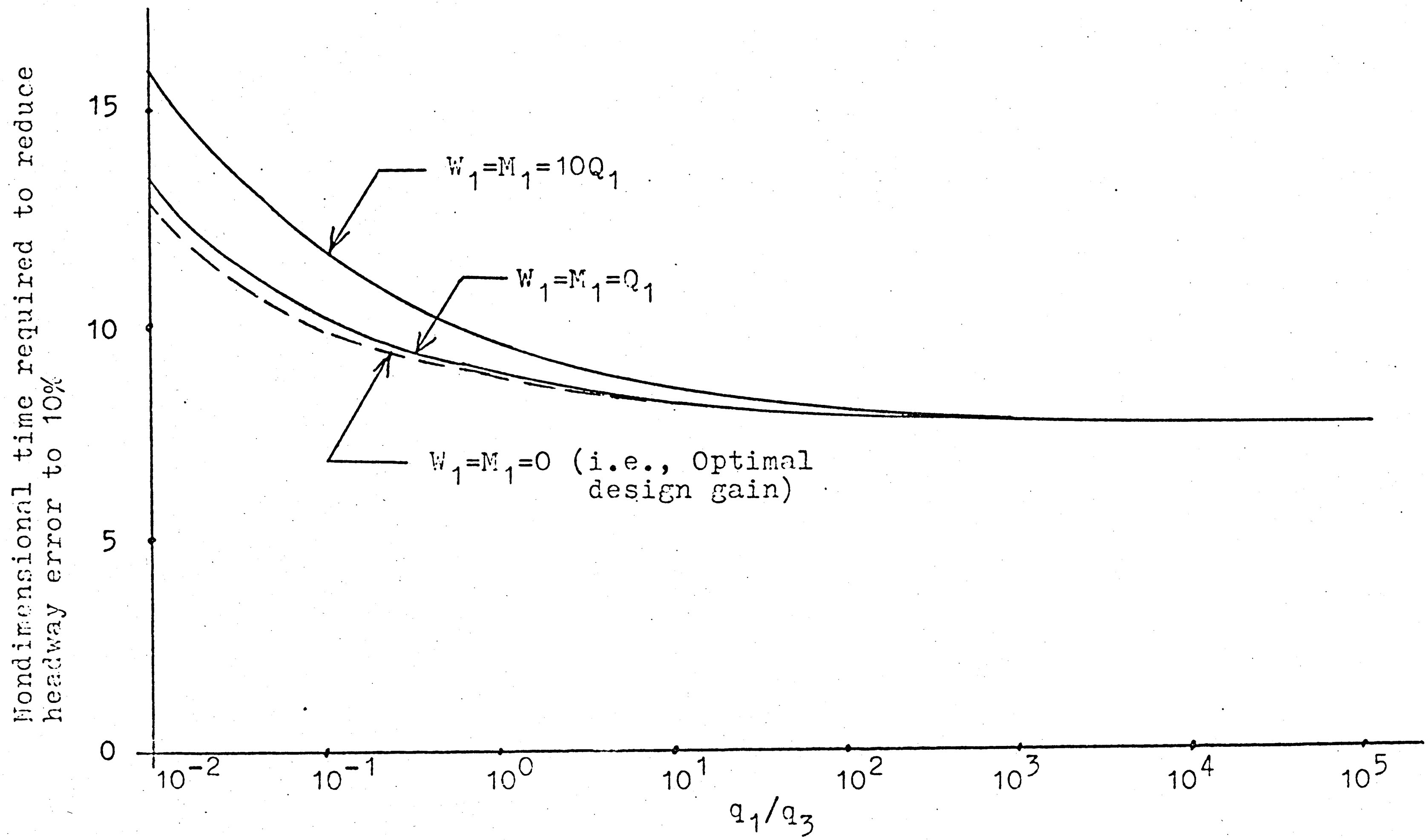


Fig.(5.1) Nondimensional time required to reduce headway error to ten percent of initial values vs. q_1/q_3 ; Q_1=diag.(q_1 , $10q_2$, 10 , 10)

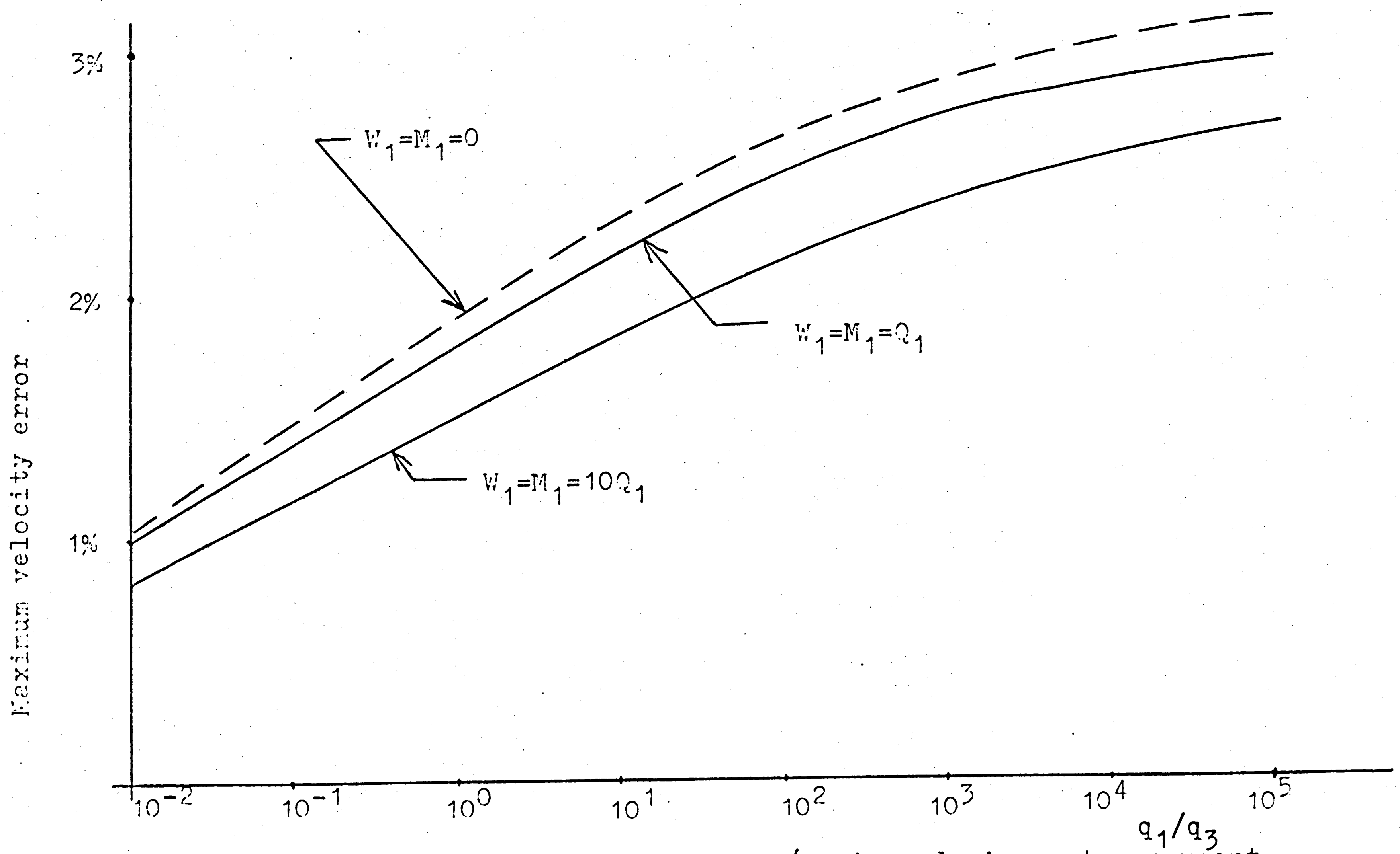


Fig.(5.2) Maximum velocity error vs. q_1/q_3 in reducing a ten percent initial headway error

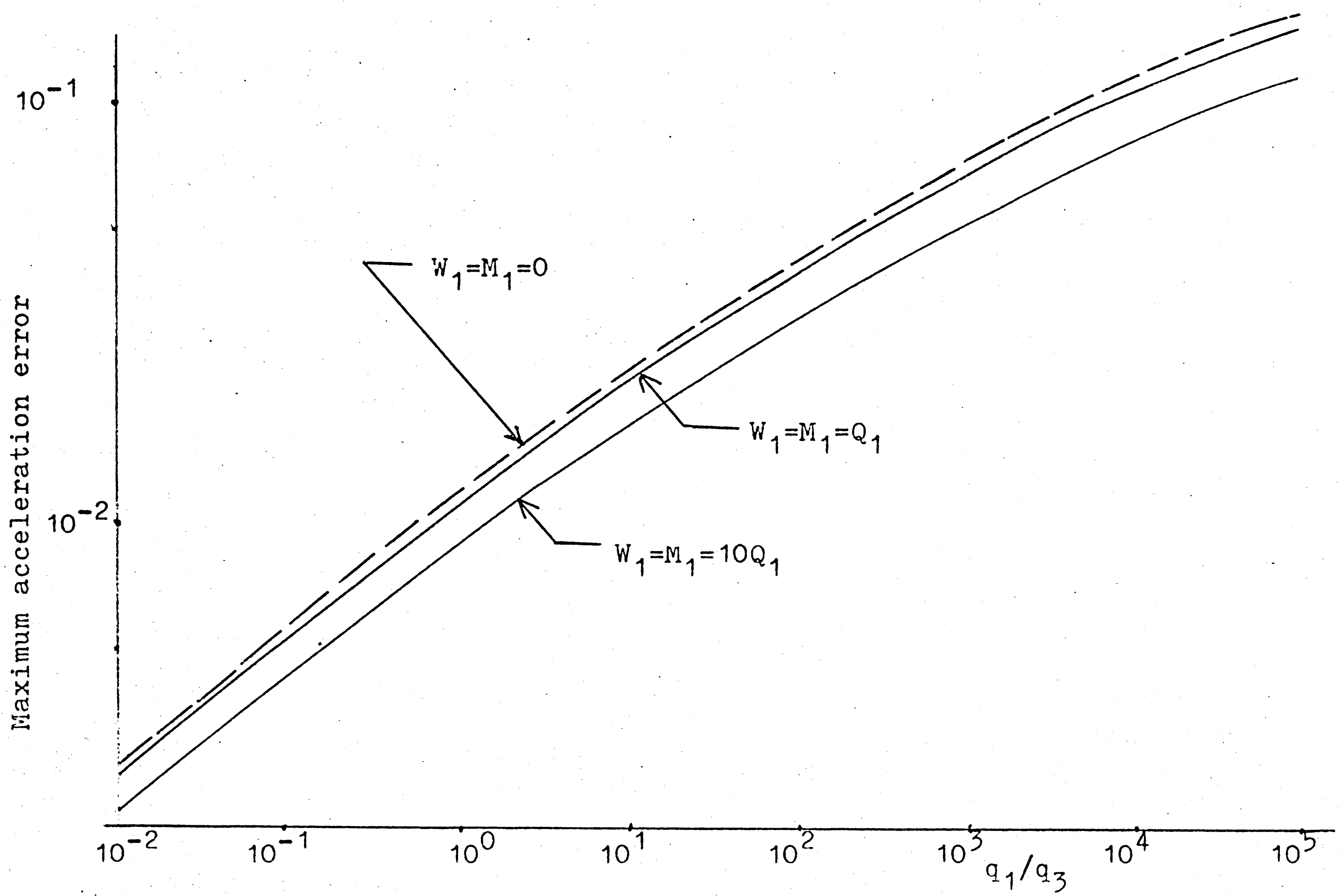


Fig.(5.3) Maximum nondimensional acceleration error vs. q_1/q_3 in reducing a ten percent initial headway error

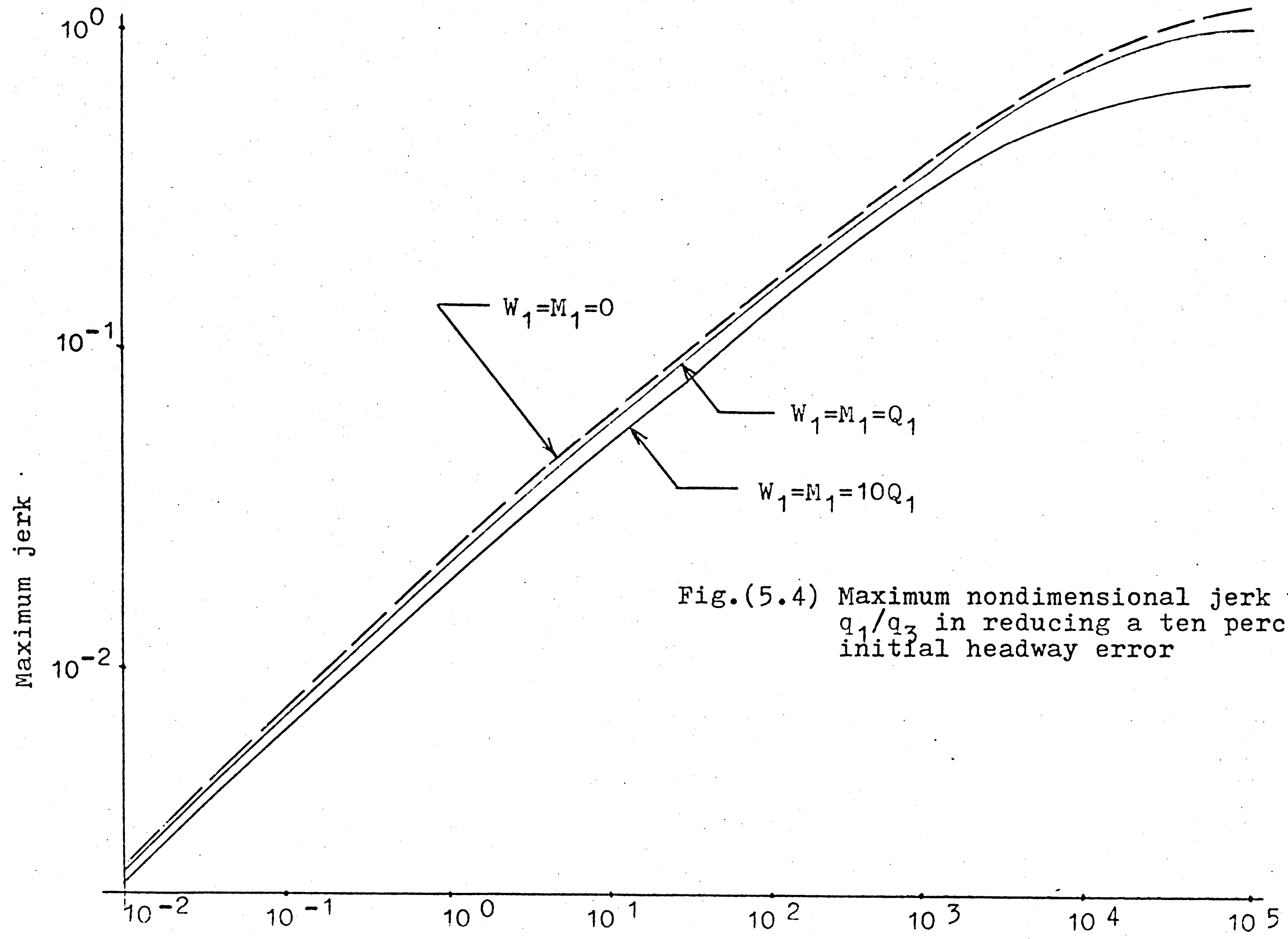


Fig.(5.4) Maximum nondimensional jerk vs. q_1/q_3 in reducing a ten percent initial headway error

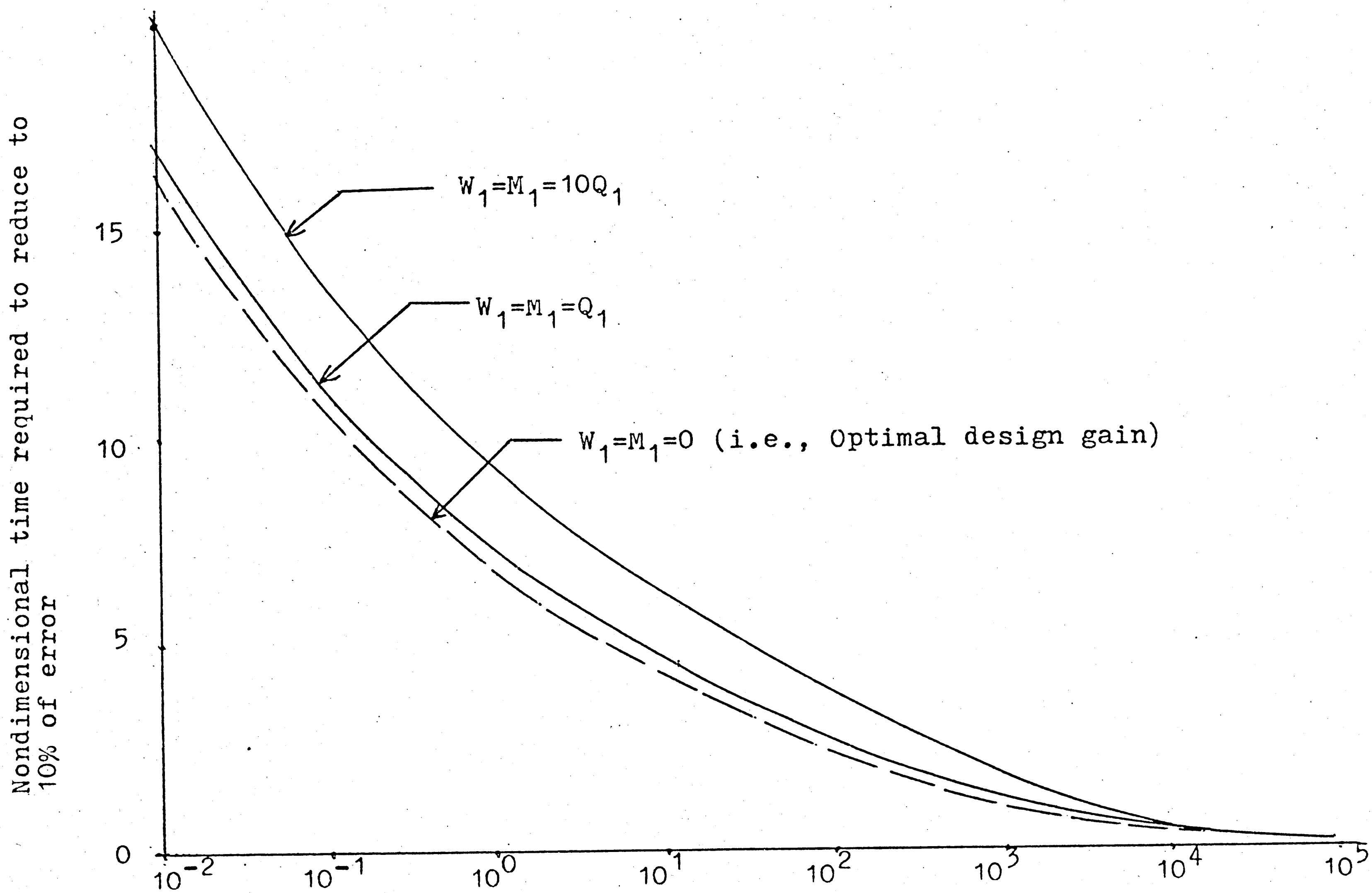


Fig.(5.5) Nondimensional time required to reduce velocity error to ten percent of initial values vs. q_1/q_3; $Q_1=\text{diag.}(q_1, 10q_1, 10, 10)$

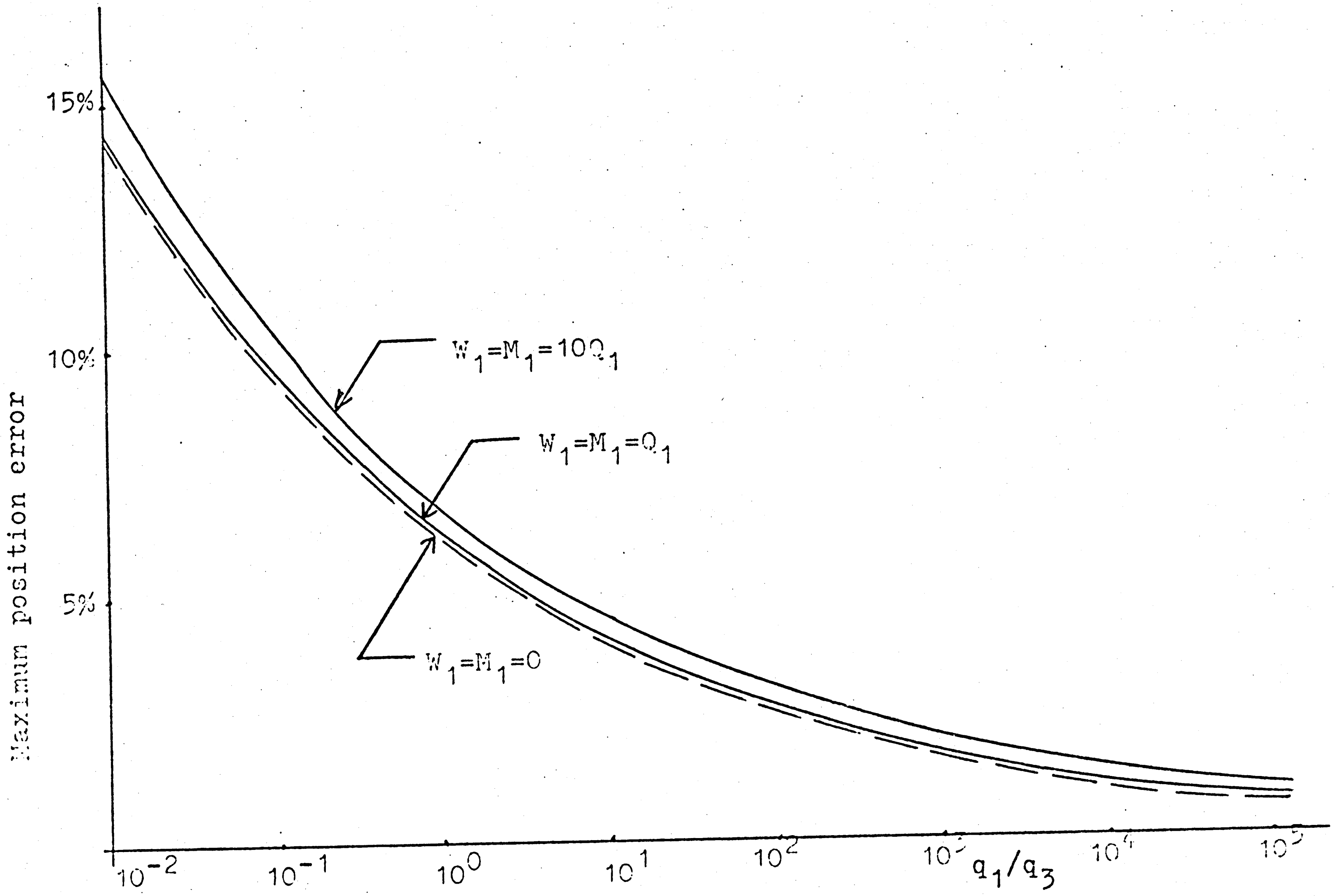


Fig.(5.6) Maximum position error vs. q_1/q_3 in reducing a five percent initial velocity error

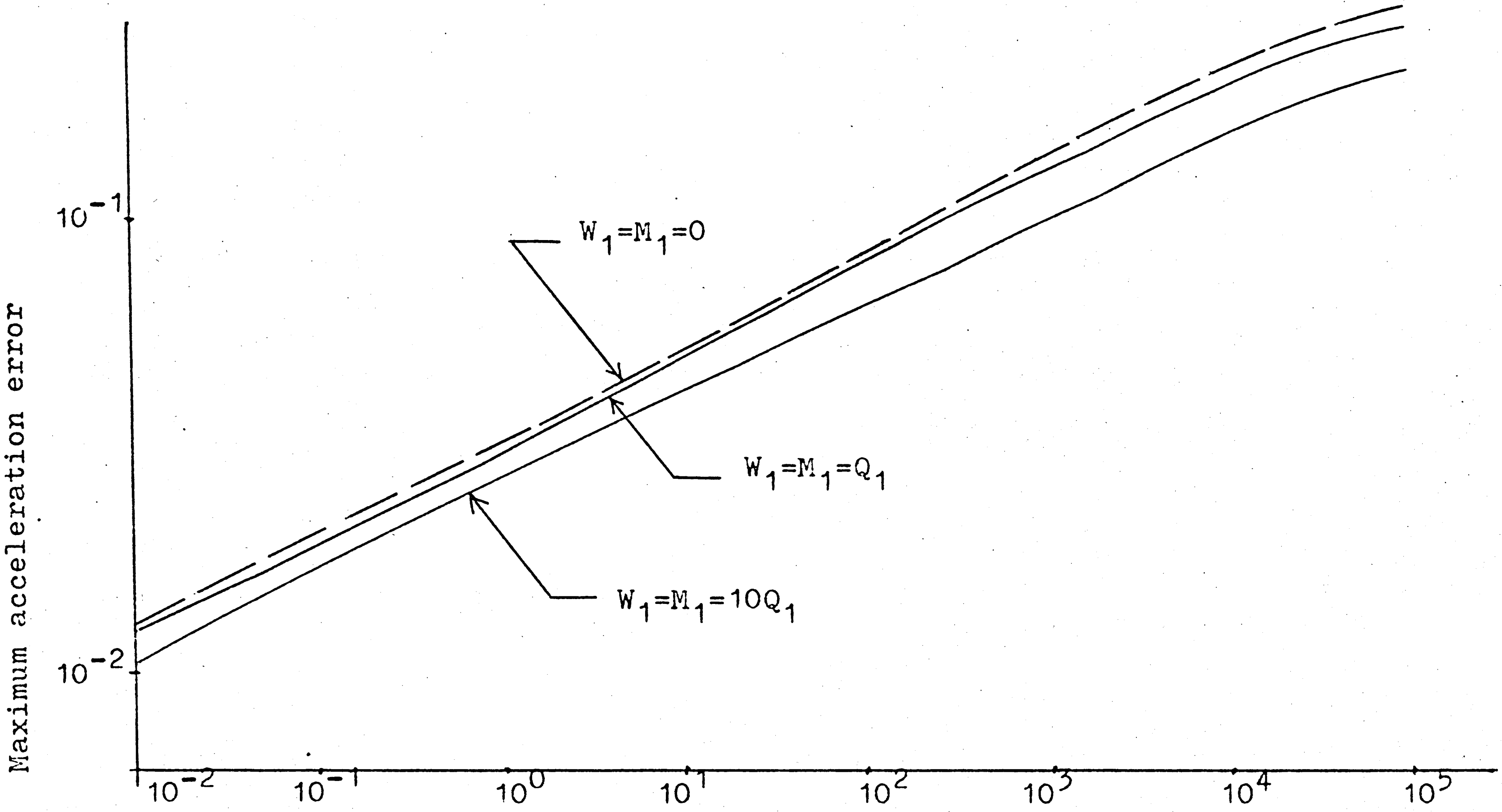
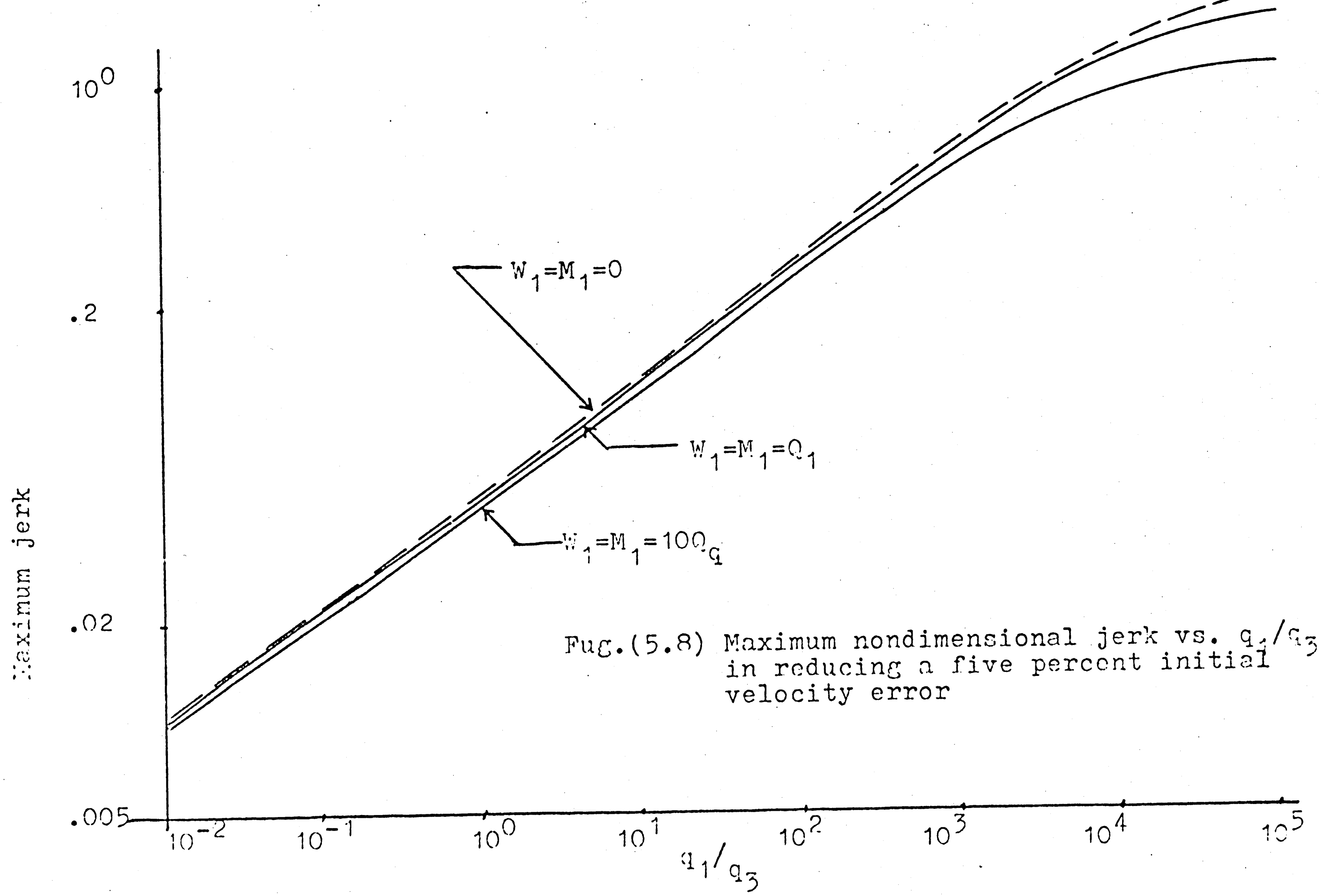


Fig.(5.7) Maximum nondimensional acceleration error vs. q_1/q_3 in reducing a five percent initial velocity error



Fug.(5.8) Maximum nondimensional jerk vs. q_1/q_3 in reducing a five percent initial velocity error the maximum jerk requirement for normal operation.

Combining all these facts, there is only one set of q_1/q_3 satisfying all these required specifications in the longitudinal control problem. That is $q_1/q_3 = 10^0$ or $q_1=q_3=10$. Thus, the weighting matrix for our design problem will be set at $Q_1=\text{diag.}(10,100,10,10)$ and the corresponding control gains for optimal design and low sensitivity design are given in Table I.

Section 5.4 Simulation Results: Deterministic Cases

The longitudinal control system was designed on the basis of linearized state equations. The choice of the control gains was done by considering only the normal mainline operation requirements. In the following section, not only normal mainline operation but also profile-following operations where the vehicle is not operating at mainline (linearized) conditions were studied.

The nonlinear vehicle dynamics (2.7) and (2.8) were simulated on a digital computer (flow chart given in Appendix C), and the following situations were studied:

(1) Mainline normal operation with an initial headway error of 5 feet for nominal case and worst cases when the variable parameters are at extreme values. [Figs.(5.9)-(5.16)]

(2) Mainline normal operation with an initial velocity error of 2.5 ft/sec for nominal case and worst cases. [Figs.(5.17)-(5.24)]

(3) Merging from an off-line station following a given trapezoidal acceleration profile. [Fig.(5.25)]

(4) Slot slipping or maneuvering following a given trapezoidal acceleration profile. [Fig.(5.26)]

(5) Emergency stopping with a constant deceleration profile. [Figs.(5.27)-(5.28)]

Section 5.4.1 Normal Mainline Operations (1),(2)

Figures (5.9)-(5.12) show the vehicle dynamic response for the optimal design for an initial position error of 5 feet. There are three trajectories shown in each figure. One is the nominal response when zero headwind and average vehicle mass. The others are worst cases when wind gust V_w=150 ft/sec, vehicle weight M=7245 lbs and V_w=-150 ft/sec, M= 2415 lbs. Simulation results show that two other worst cases (V_w=150 ft/sec, M=2415 lbs and V_w=-150 ft/sec, M=7245 lbs) are not so severe and thus they will not be shown in the figure. Fig.(5.9) shows excellent position error response which is insensitive to parameter variations. Fig.(5.10) shows good velocity error response which is somewhat sensitive to parameter variations. Figs.(5.11)-(5.12) show fair acceleration and jerk response which are very

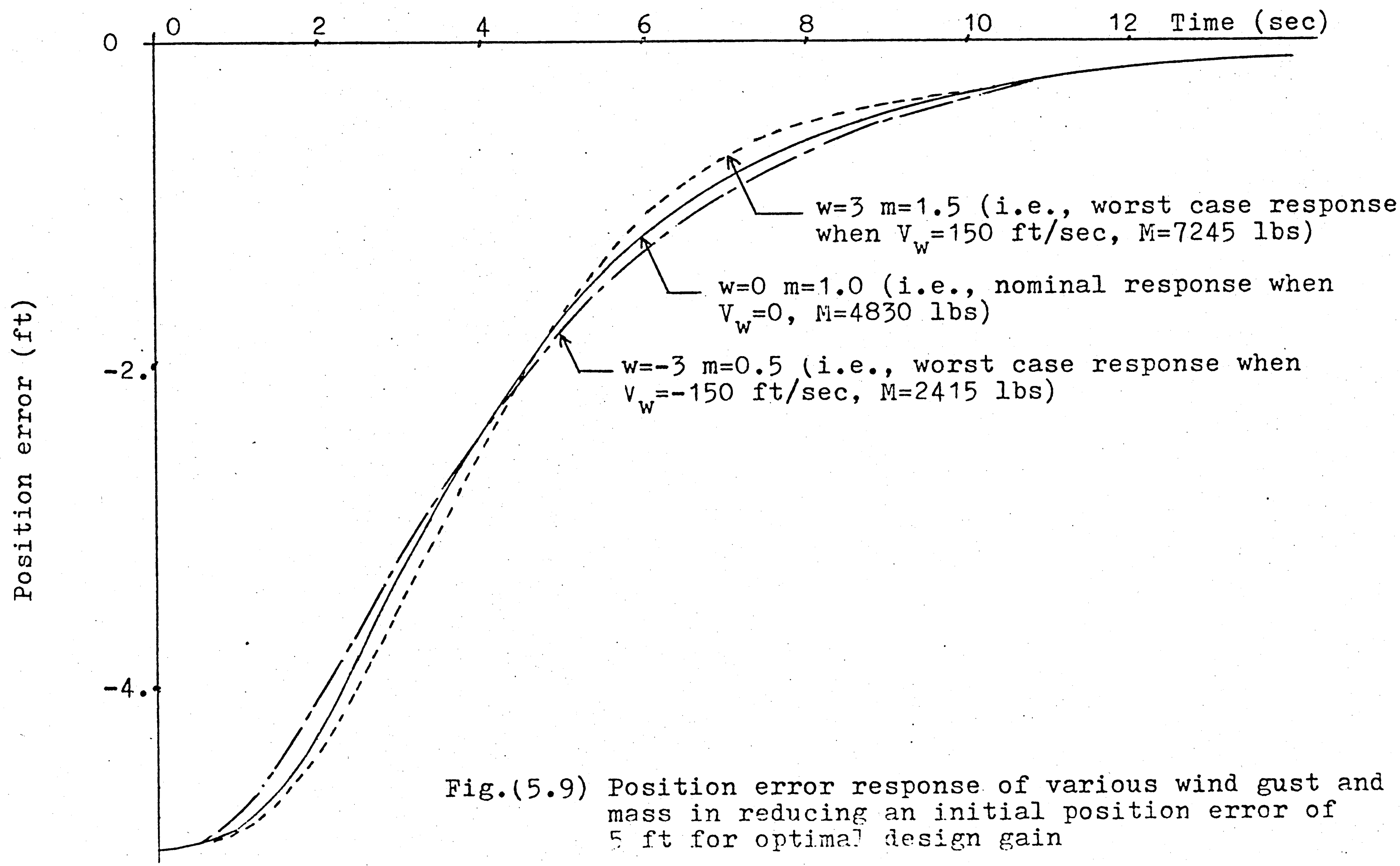


Fig.(5.9) Position error response of various wind gust and mass in reducing an initial position error of 5 ft for optimal design gain

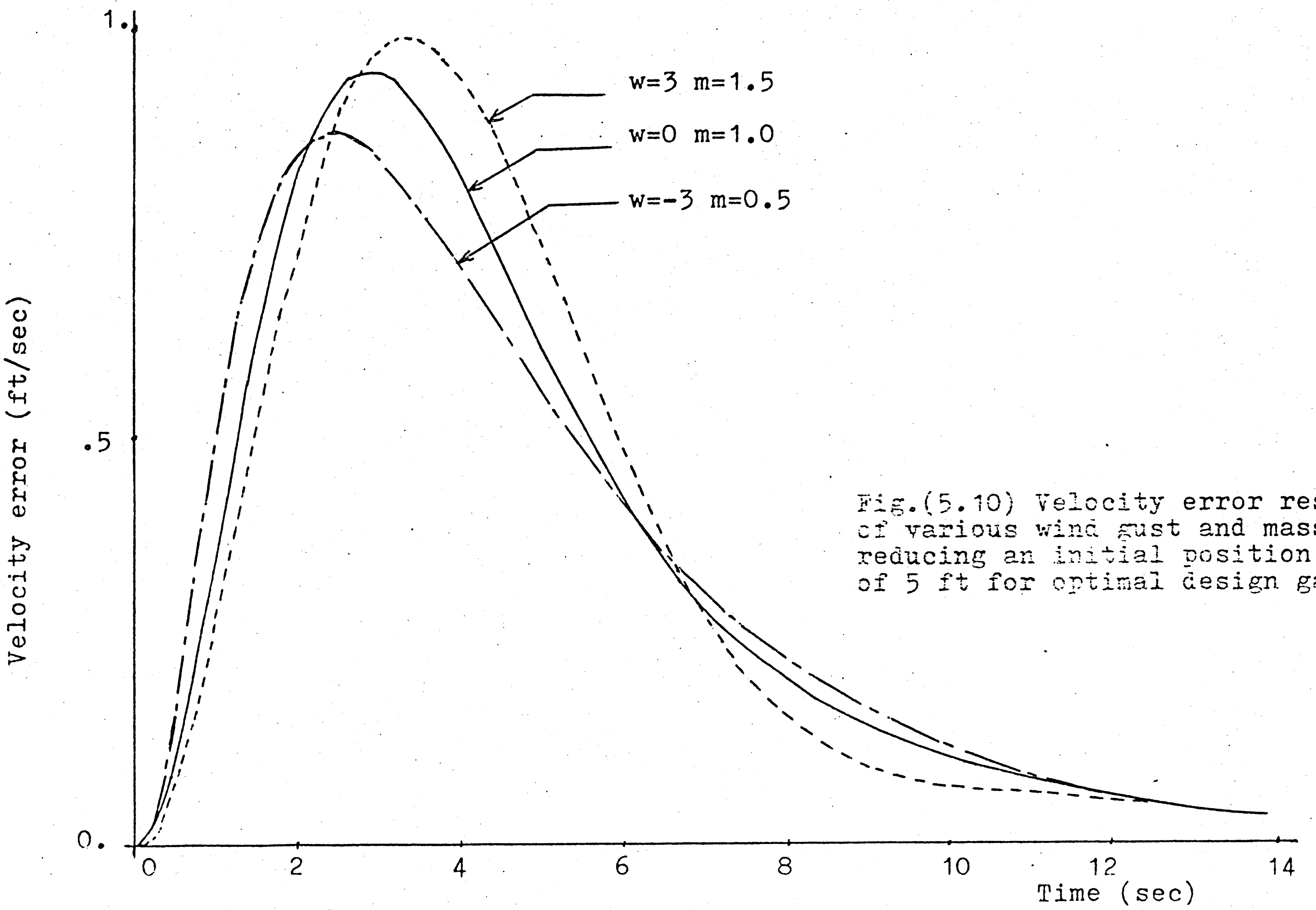


Fig.(5.10) Velocity error response of various wind gust and mass in reducing an initial position error of 5 ft for optimal design gain

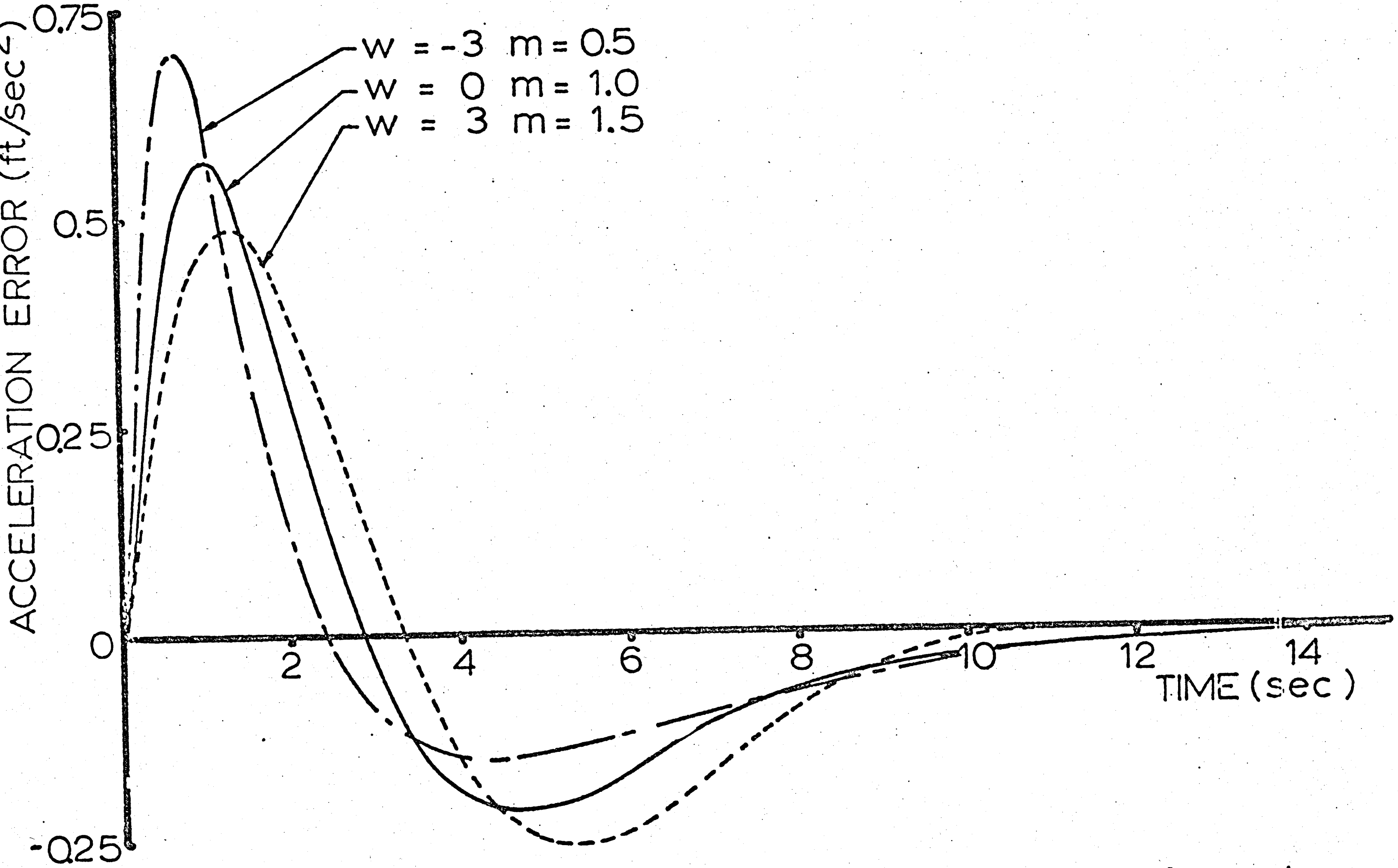


Fig.(5.11) Acceleration response of various wind gust and mass in reducing an initial position error of 5ft for optimal design gain

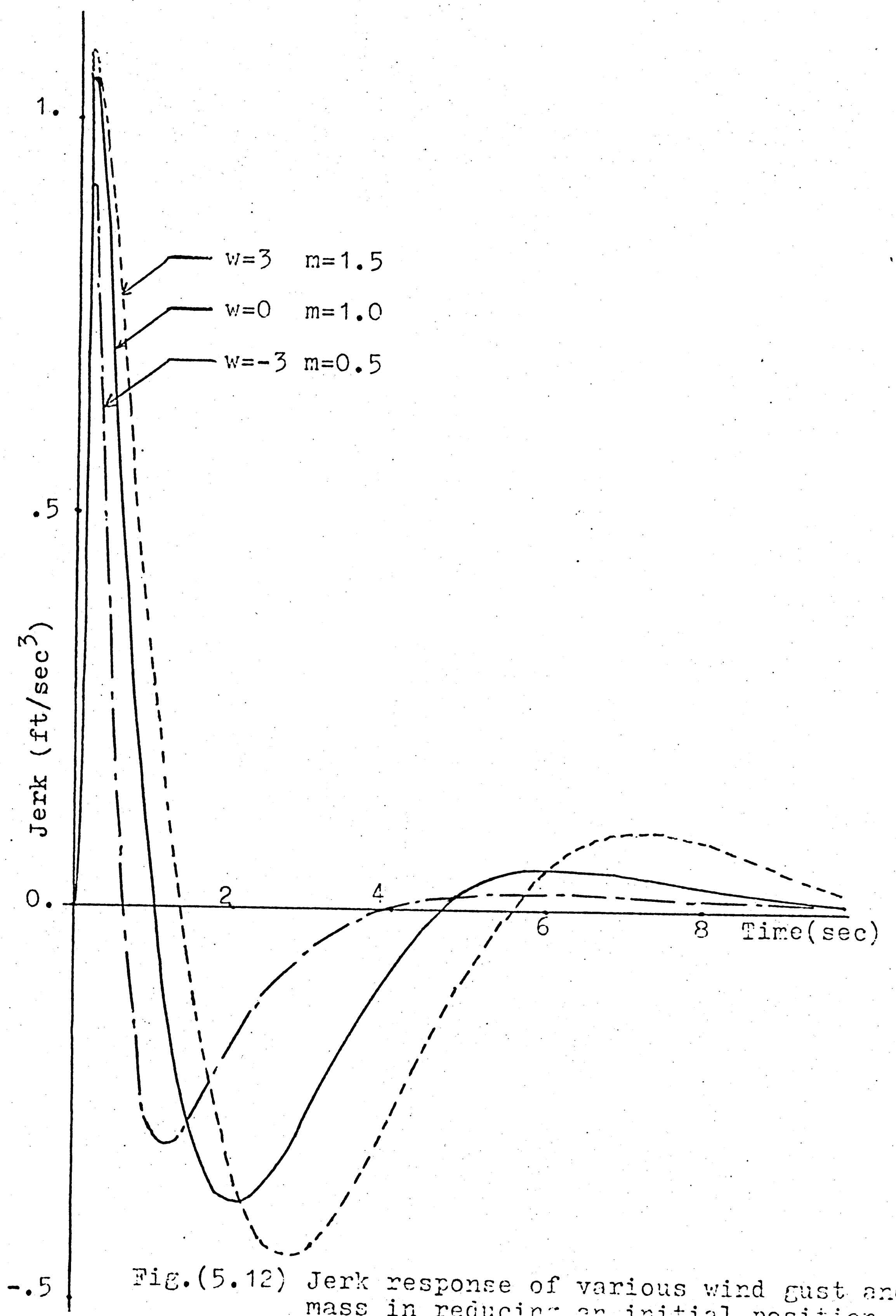


Fig.(5.12) Jerk response of various wind gust and mass in reducing an initial position error of 5ft for optimal design gain sensitive to parameter variations.

Figures (5.13)-(5.16) are the vehicle dynamic response for the low sensitivity design gain $W_1=M_1=10Q_1$ under the same initial condition as Figs.(5.9)-(5.12). The position error is not much different from the optimal design as seen from Fig.(5.13). The velocity error response is also little changed. However, the jerk and acceleration responses are reduced greatly and are relatively insensitive to parameter variations as seen from Figs.(5.15)-(5.16).

Figures (5.17)-(5.24) show the vehicle dynamic response for an initial velocity error of 2.5 ft/sec for the optimal design gain and the low sensitivity design gain $W_1=M_1=10Q_1$. By comparison, it is observed that the sensitivity of dynamic response to parameter variations and maximum values of the error variables are reduced. This reduction is especially noticable in the acceleration and jerk responses.

Section 5.4.2 Profile-Following Operations (3)-(5)

During merging, maneuvering, and emergency stopping the vehicle is not operated at the nominal conditions. For example, in merging from an off-line station the vehicle starts zero velocity and accelerates to line velocity following a commanded acceleration profile as shown in Fig.(5.25). The differential equations

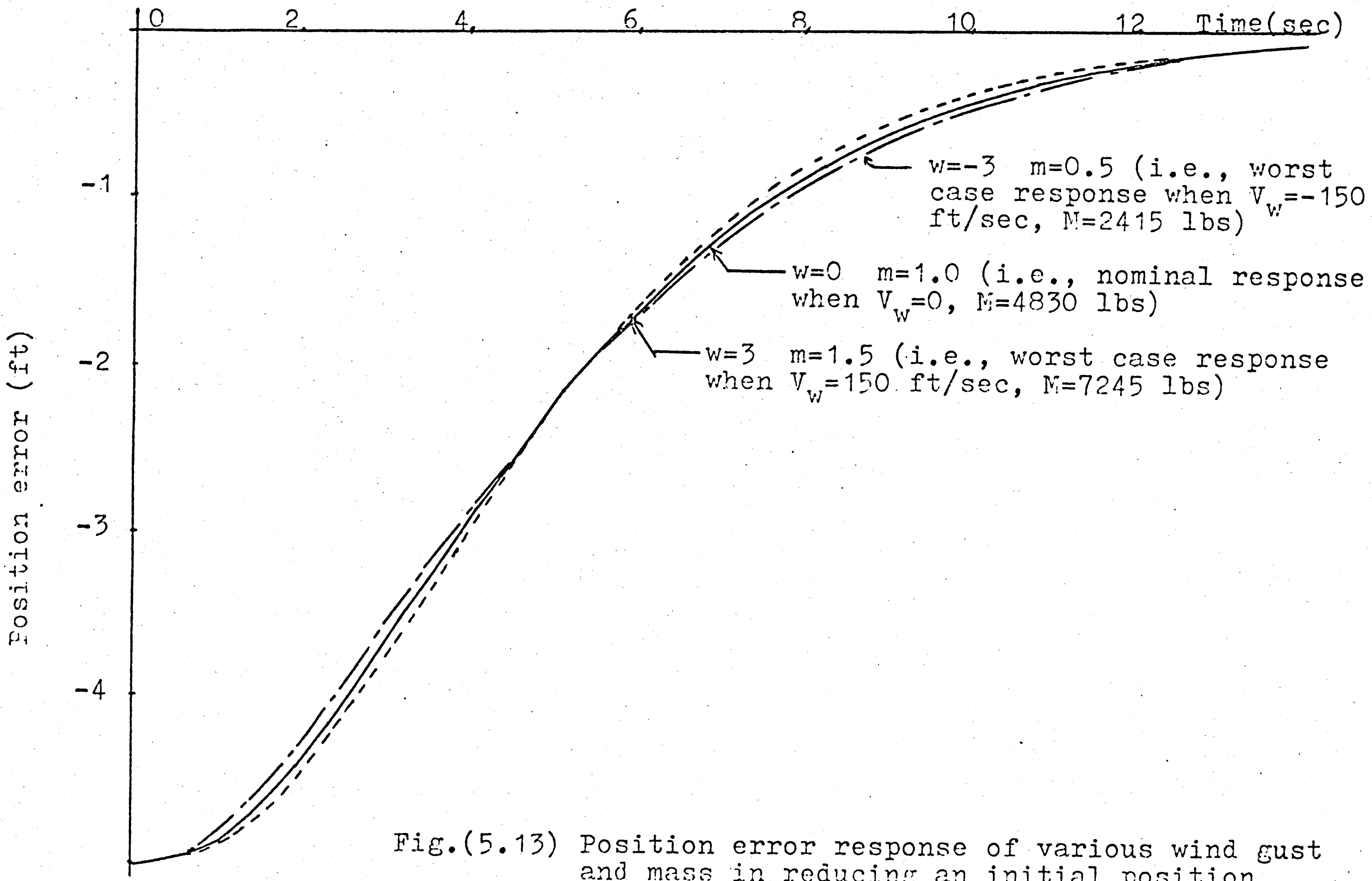


Fig.(5.13) Position error response of various wind gust and mass in reducing an initial position error of 5ft for low sensitivity design gain

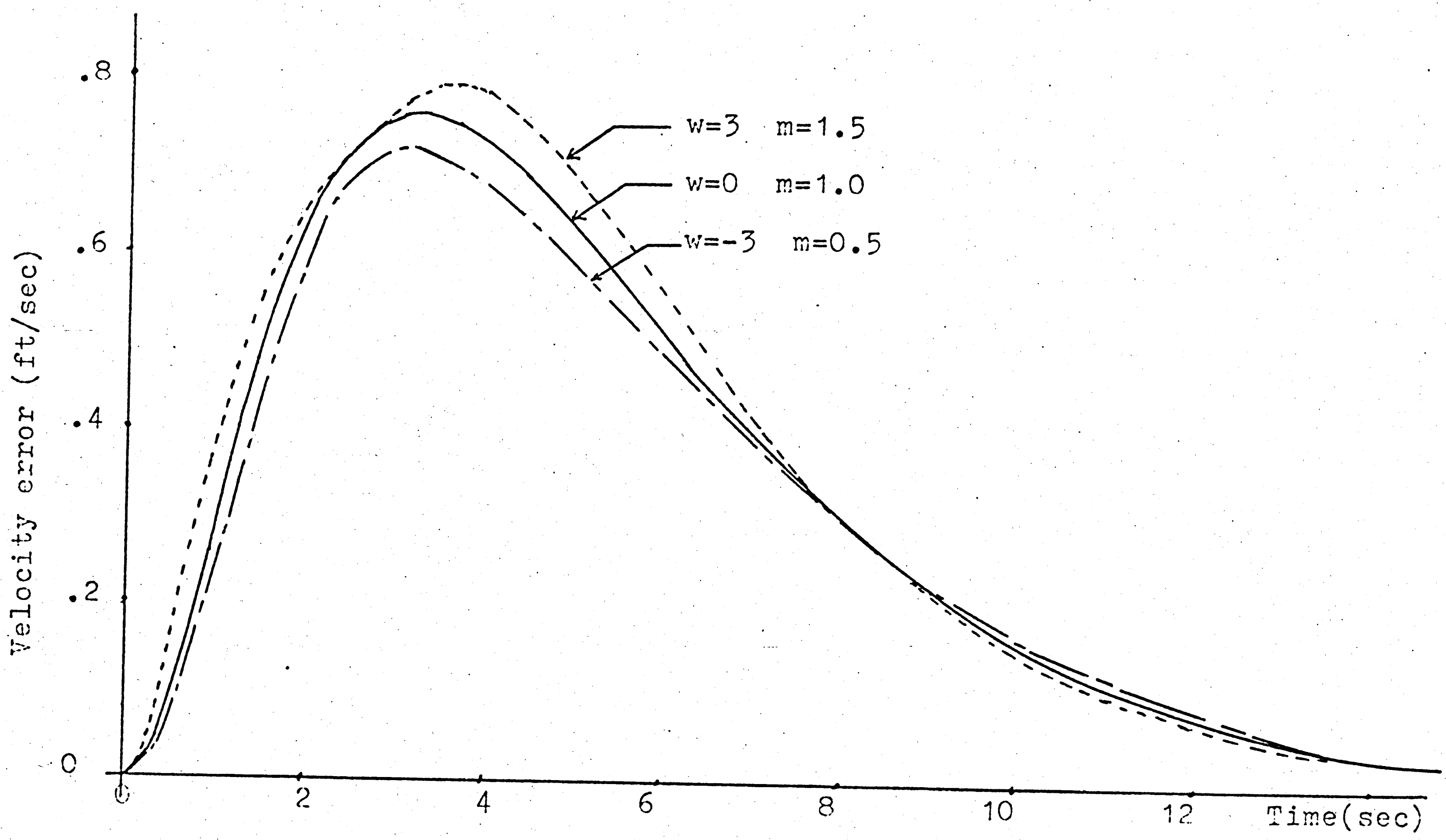


Fig.(5.14) Velocity error response of various wind gust and mass in reducing an initial position error of 5ft for low sensitivity design gain

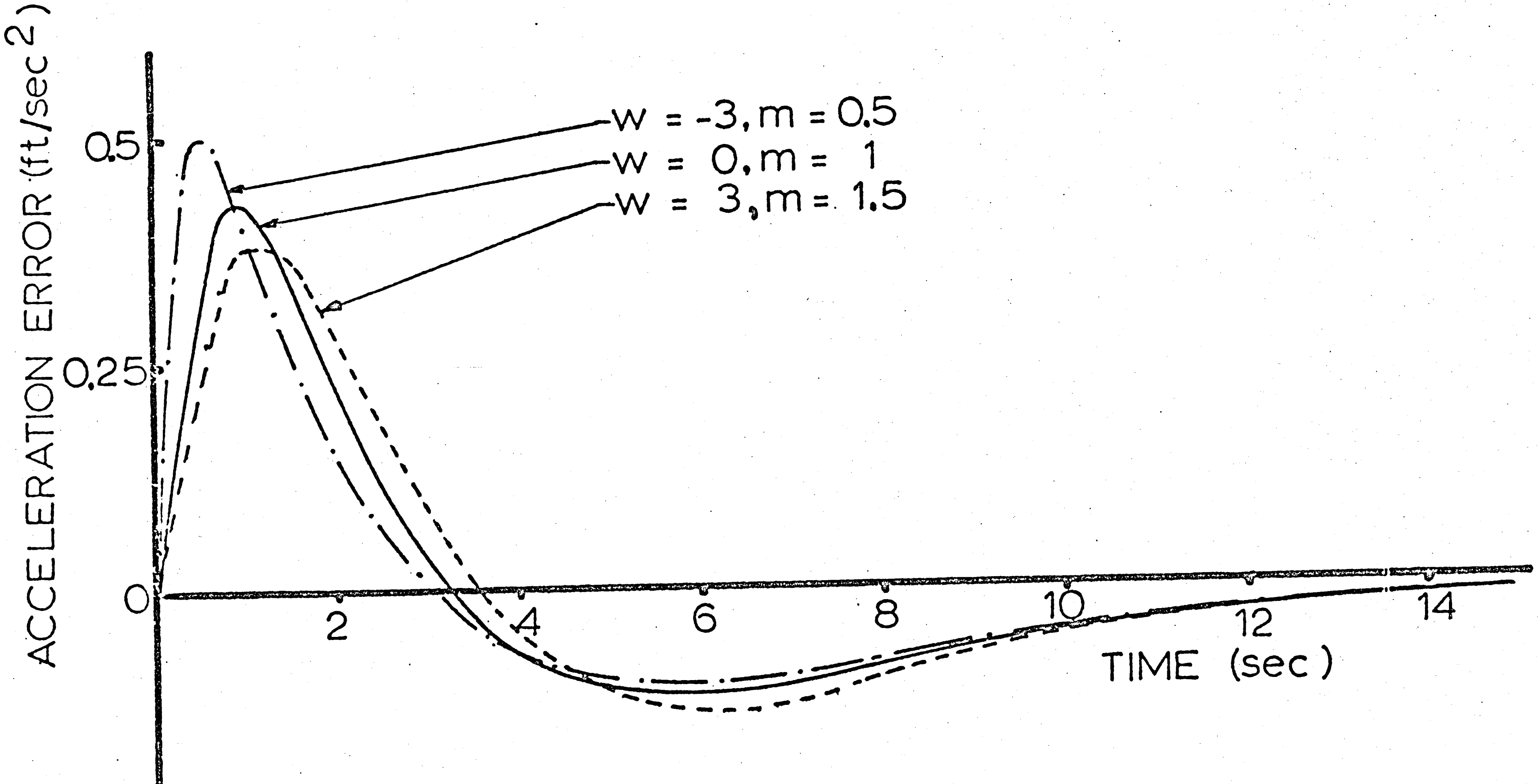


Fig.(5.15) Acceleration error response of various wind gust and mass in reducing an initial position error of 5ft for low sensitivity design gain

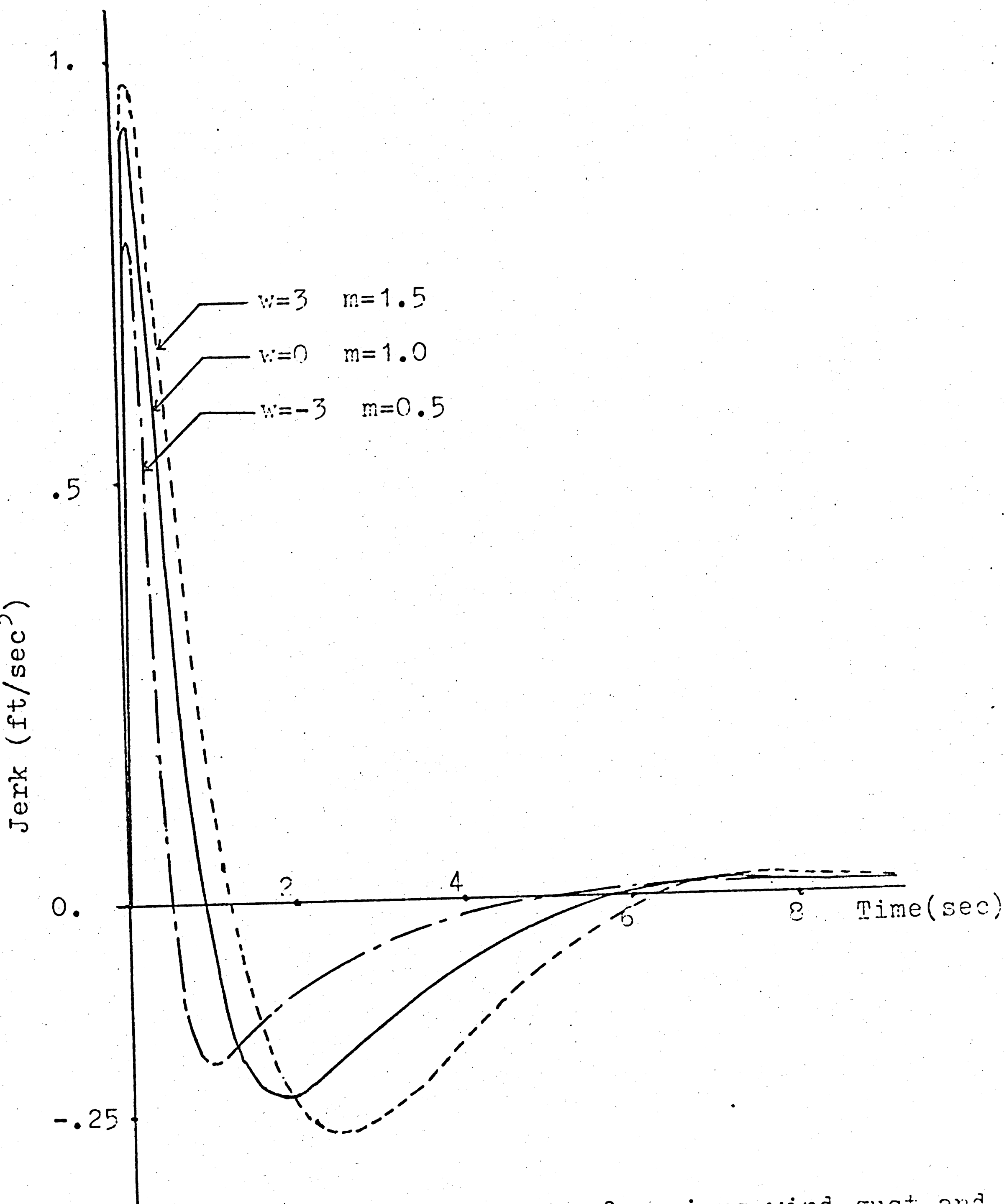


Fig.(5.16) Jerk response of various wind gust and mass in reducing an initial position error of 5ft for low sensitivity design gain

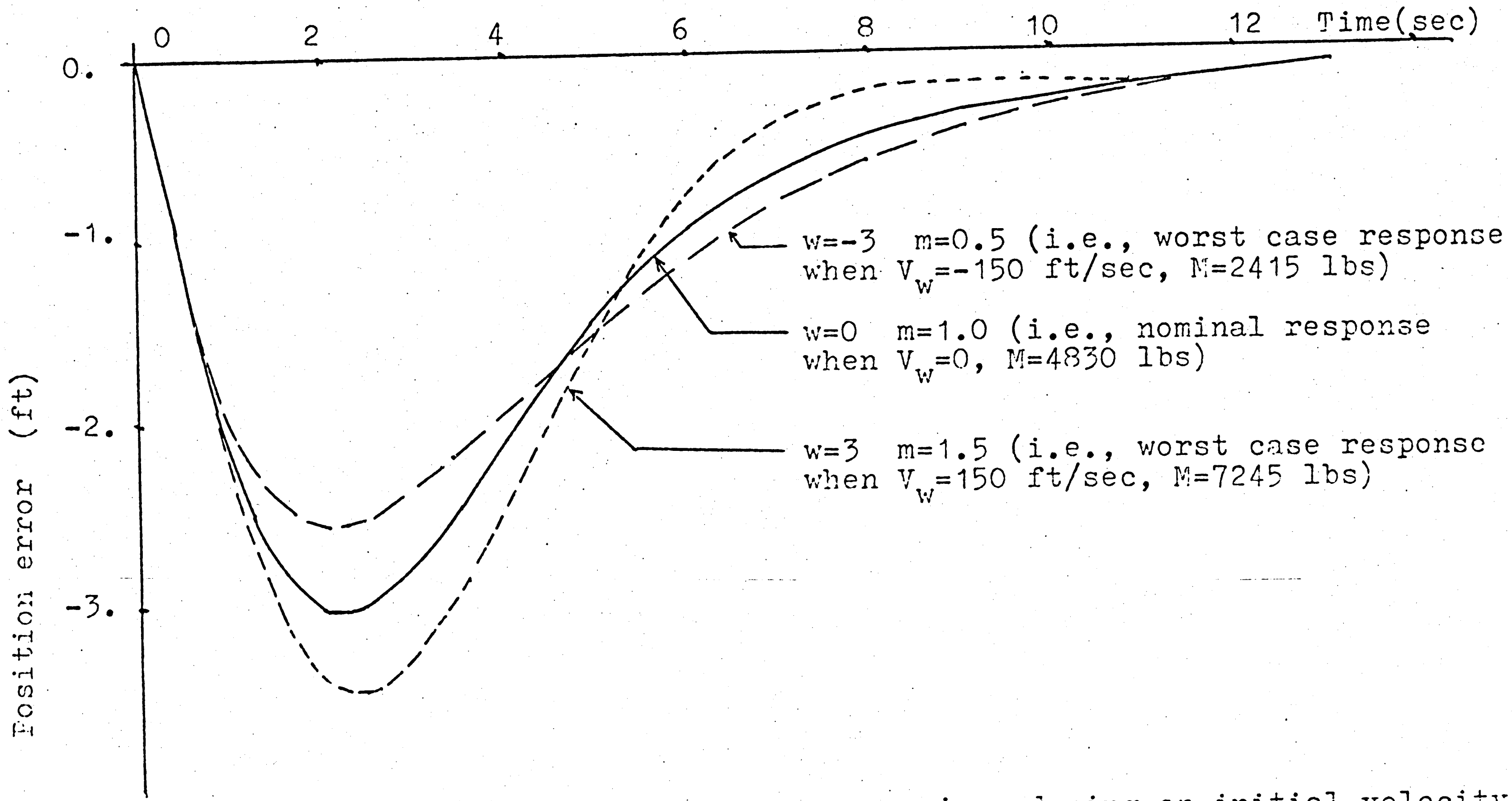


Fig.(5.17) Position error response in reducing an initial velocity error of five percent to normal state using optimal design gain

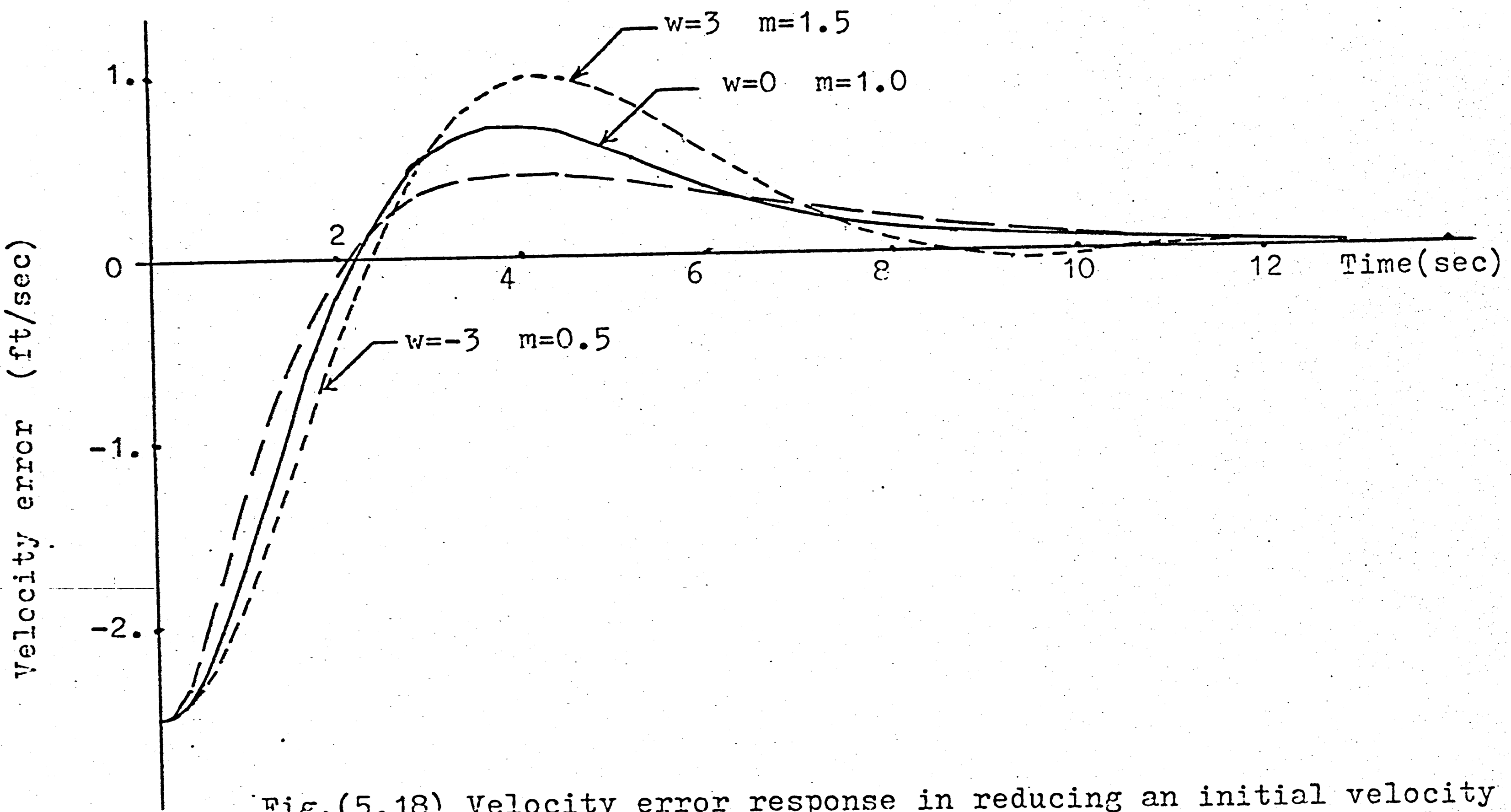


Fig.(5.18) Velocity error response in reducing an initial velocity error of five percent to normal states using optimal design gain

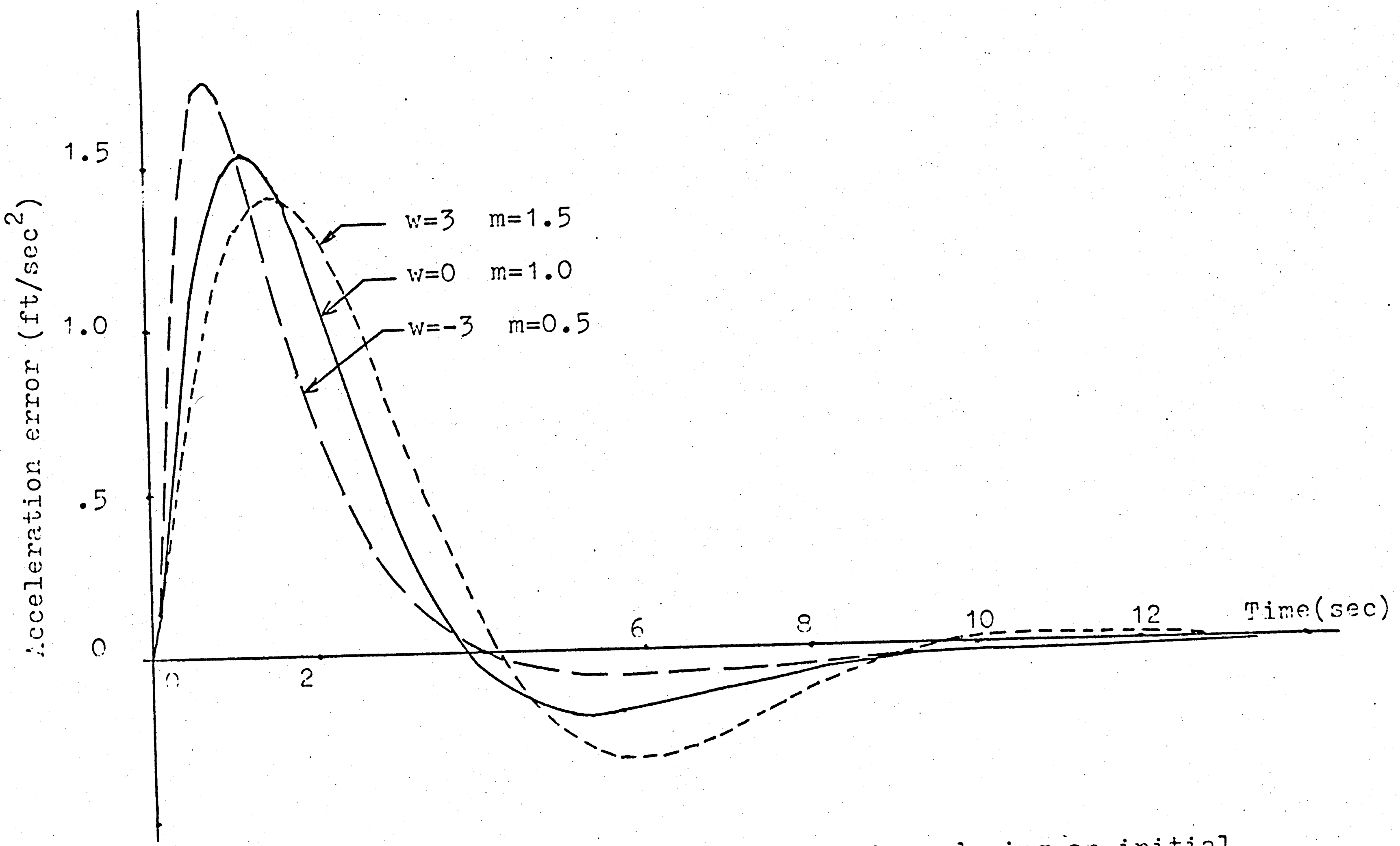


Fig.(5.19) Acceleration error response in reducing an initial velocity error of five percent to normal states using optimal design gain

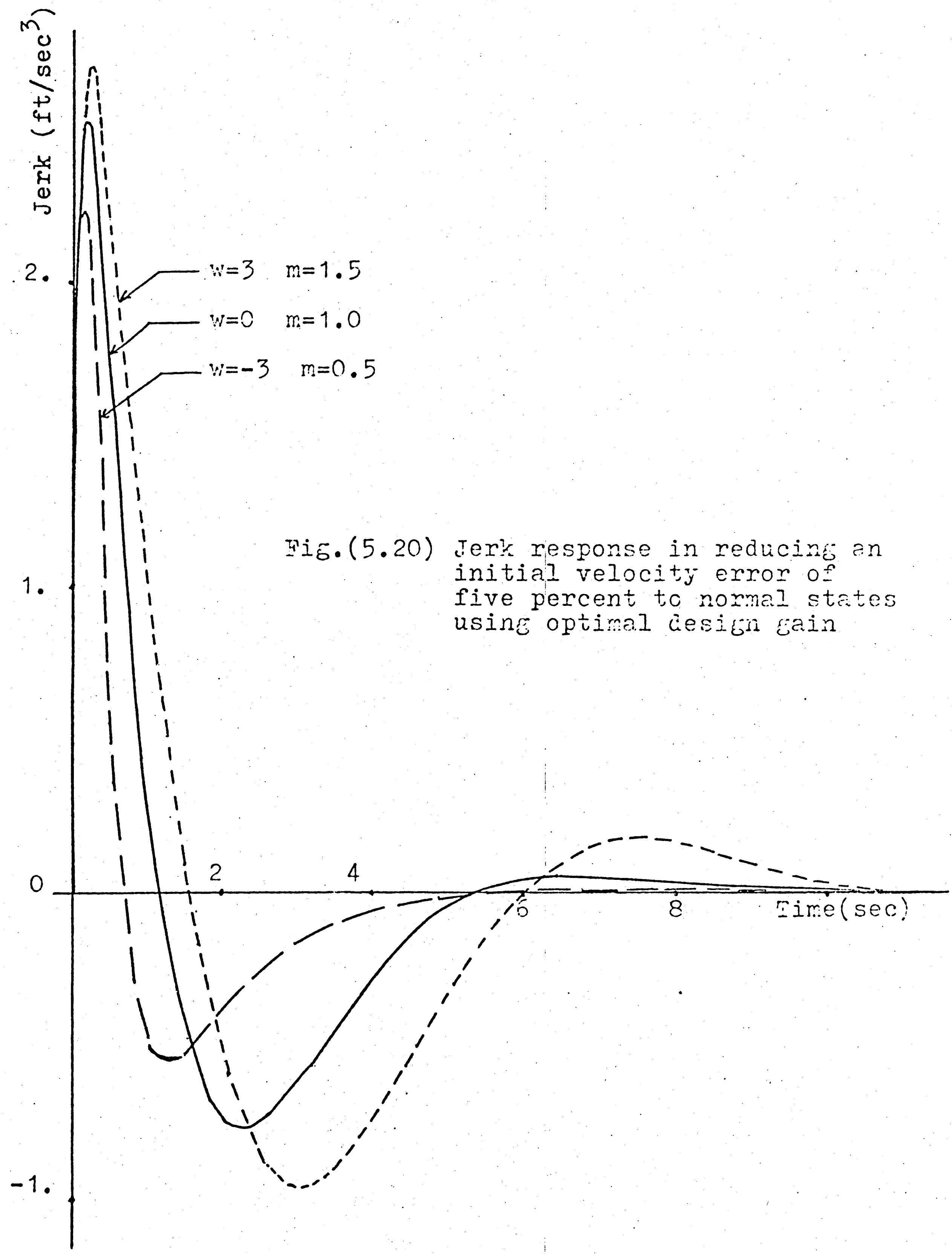


Fig.(5.20) Jerk response in reducing an initial velocity error of five percent to normal states using optimal design gain

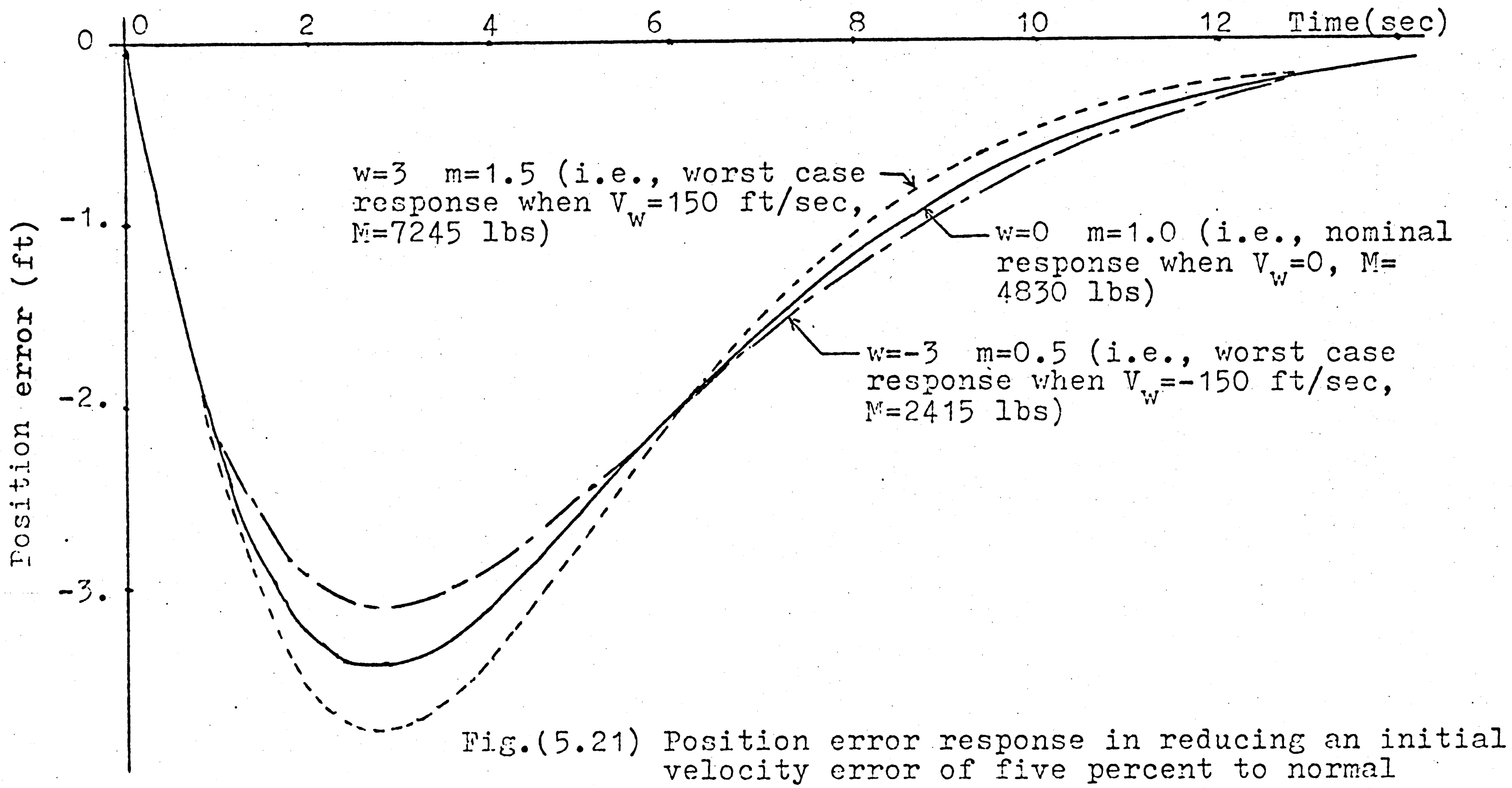


Fig.(5.21) Position error response in reducing an initial velocity error of five percent to normal states using low sensitivity design gain $10Q_1$

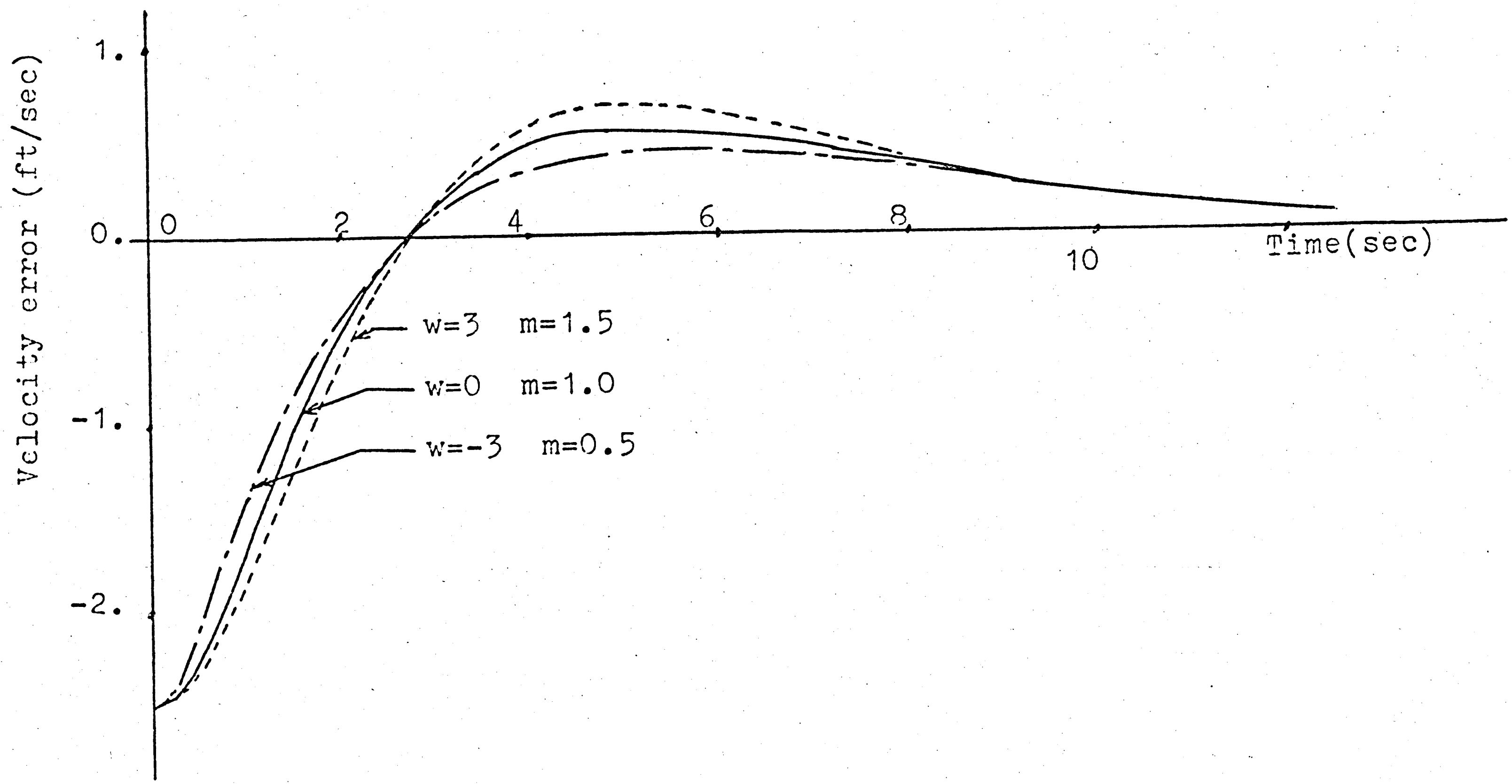


Fig.(5.22) Velocity error response in reducing an initial velocity error of five percent to normal states using low sensitivity design gain $10Q_1$

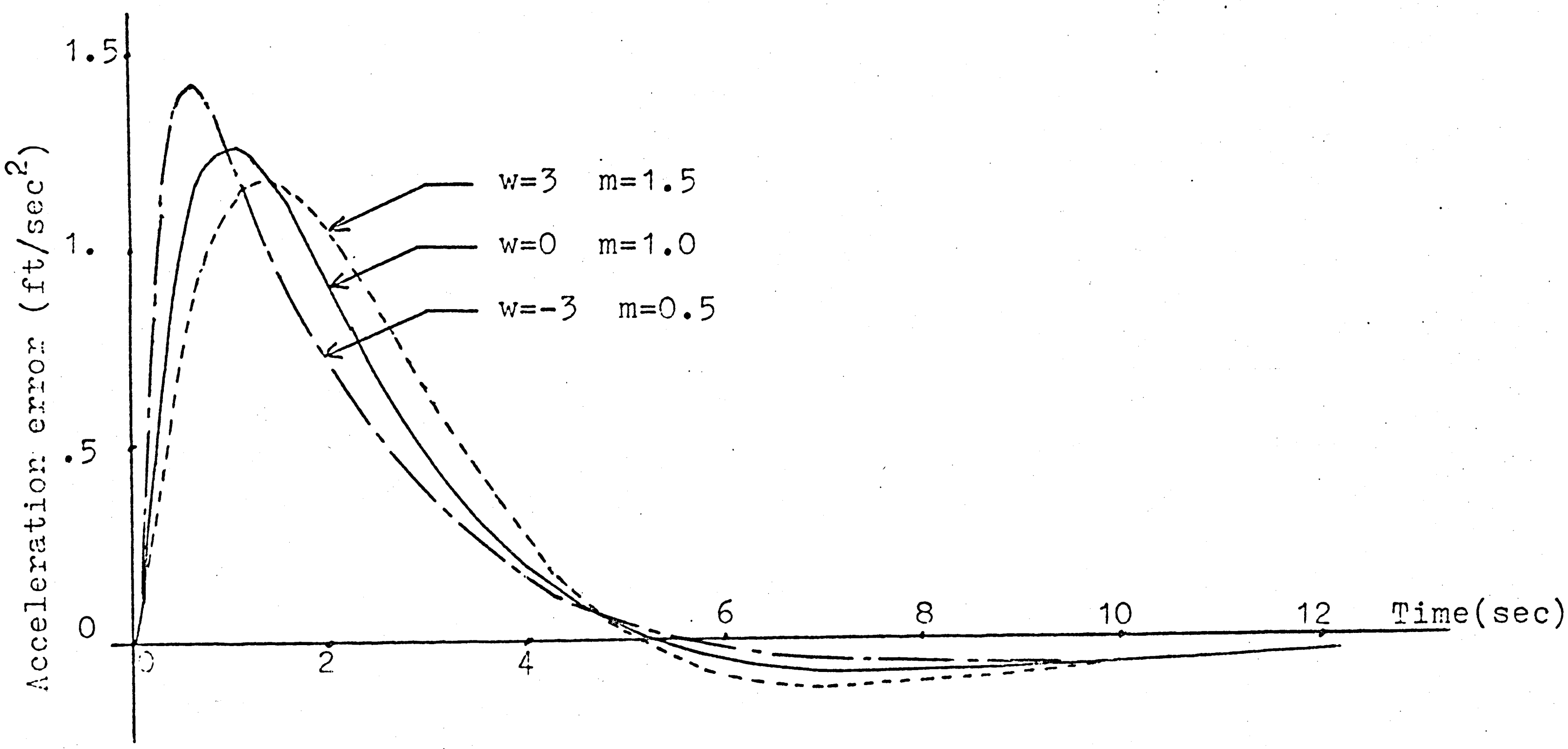


Fig.(5.23) Acceleration error response in reducing an initial velocity error of five percent to normal states using low sensitivity design gain $10Q_1$

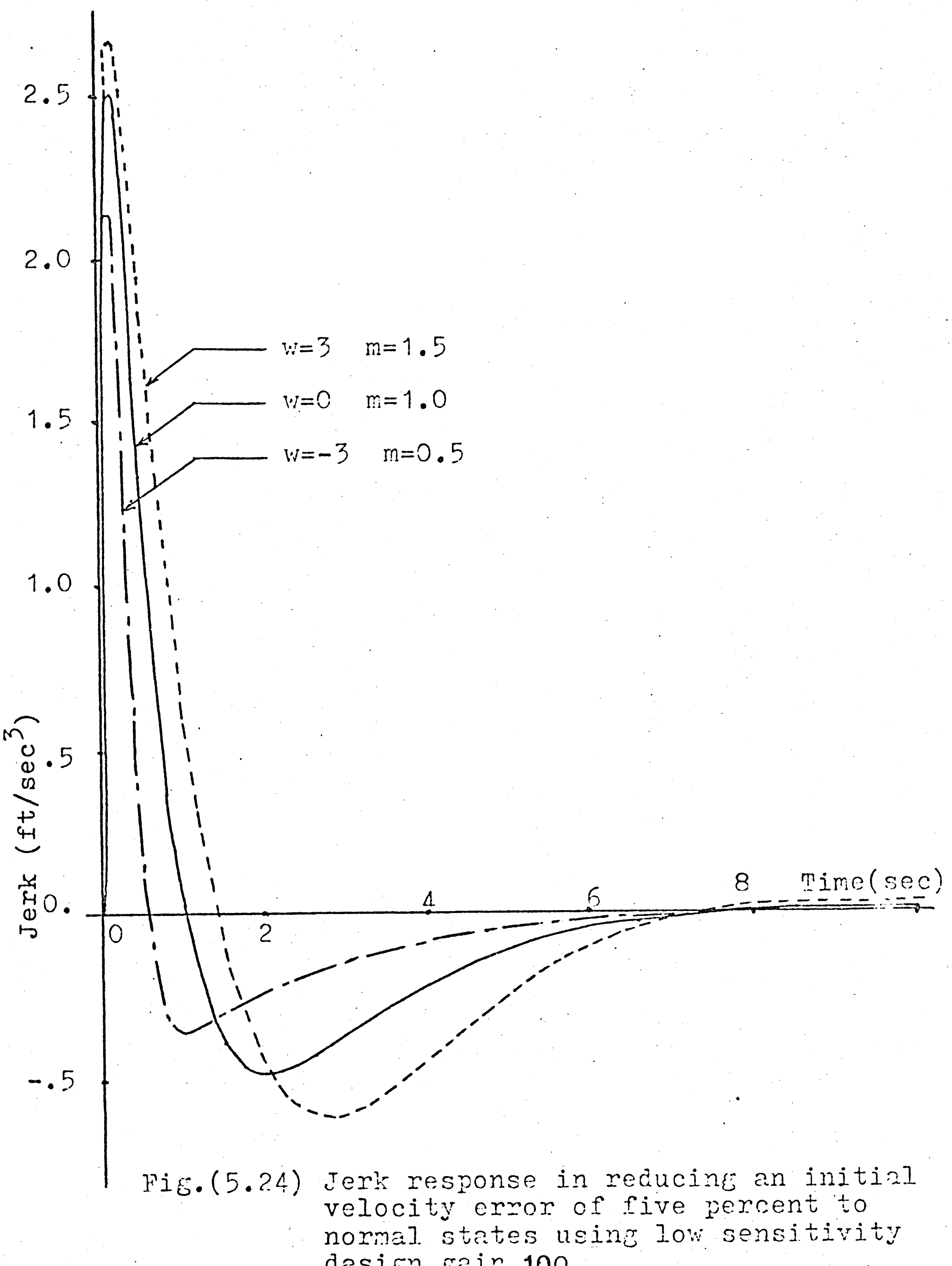


Fig.(5.24) Jerk response in reducing an initial velocity error of five percent to normal states using low sensitivity design gain $10Q_1$

(2.7) and (2.8) describing the state of the control system are nonlinear with time-varying coefficients. The longitudinal controller in this study was designed on the basis of linearized state equations; however, as seen in Fig.(5.25), the resulting controller follows the desired profiles very well even though the conditions are vastly different from those encountered during mainline operation. This is important since it shows that a linear controller with fixed gains is sufficient for various control operations and thus the complexity of the control system is minimal. In the following the values for maximum commanded acceleration were calculated from given acceleration profiles with specified time interval.

Determining the Maximum Commanded Acceleration for Merging

Assume that it be desired to merging the vehicle at exactly 10 seconds following the given acceleration profile with the maximum commanded acceleration at a_c ft/sec^2 as shown in Fig.(5.25). From simple calculations,

$$\text{commanded velocity } \left(\text{at } t=10 \text{ sec}\right) = 9a_c$$

$$\text{commanded position } \left(\text{at } t=10 \text{ sec}\right) = 45a_c$$

Since the mainline velocity is 50 ft/sec, the maximum commanded acceleration, a_c, should be 5.556 ft/sec^2. The acceleration lane length is 250 feet,

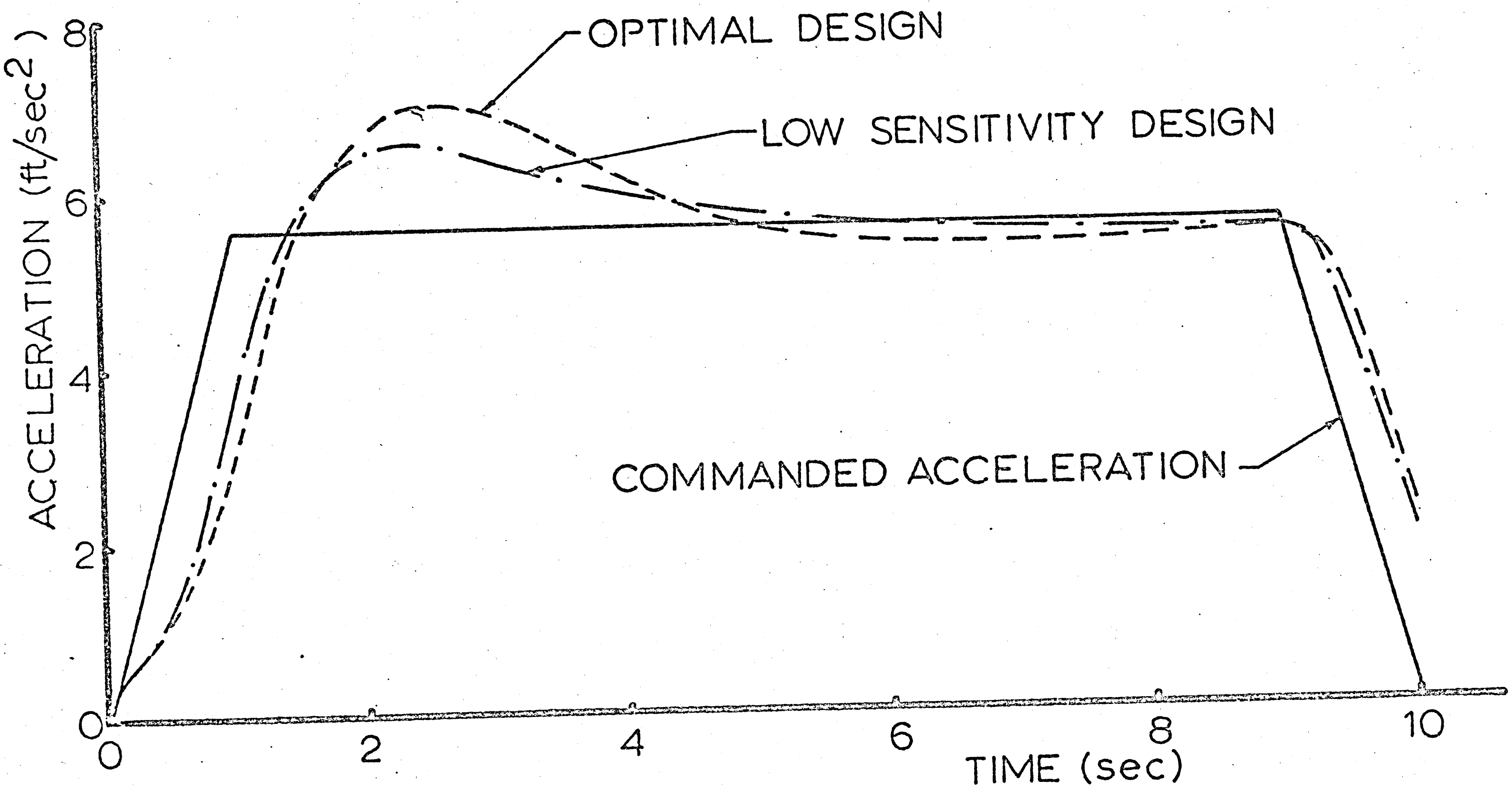


Fig.(5.25) Vehicle acceleration response during merging from an off-line station for various designs which coincides with a formula derived by Dais. [D2]

Another method which specified the maximum commanded acceleration for given acceleration profile has been investigated in Ref. [G1]. However, the time interval required to achieve merging or maneuvering is not necessary an integer.

<u>Determining the Maximum Commanded Acceleration for Slot-slipping</u>

Suppose that it be desired to slip the vehicle one slot backwards at exactly 14 seconds following the given trapezoidal acceleration profile with the maximum acceleration at $\pm a_c$ ft/sec^2 as shown in Fig.(5.26). At time $t=14$ sec, it is found that

commanded velocity = 50 ft/sec

commanded position = $(700-42a_c)$ ft

Since one slot length is 50 feet, the value of a_c should be 1.19 ft/sec^2.

Using these maximum commanded acceleration values, the results of merging and maneuvering operations are summarized in Tables II, III. Figs.(5.25)-(5.26) show the vehicle acceleration responses during merging and slot-slipping, respectively. It is seen that the responses follow the desired profile in an excellent manner and the low sensitivity design gain $W_1=M_1=10Q_1$ has the smoother response than that of

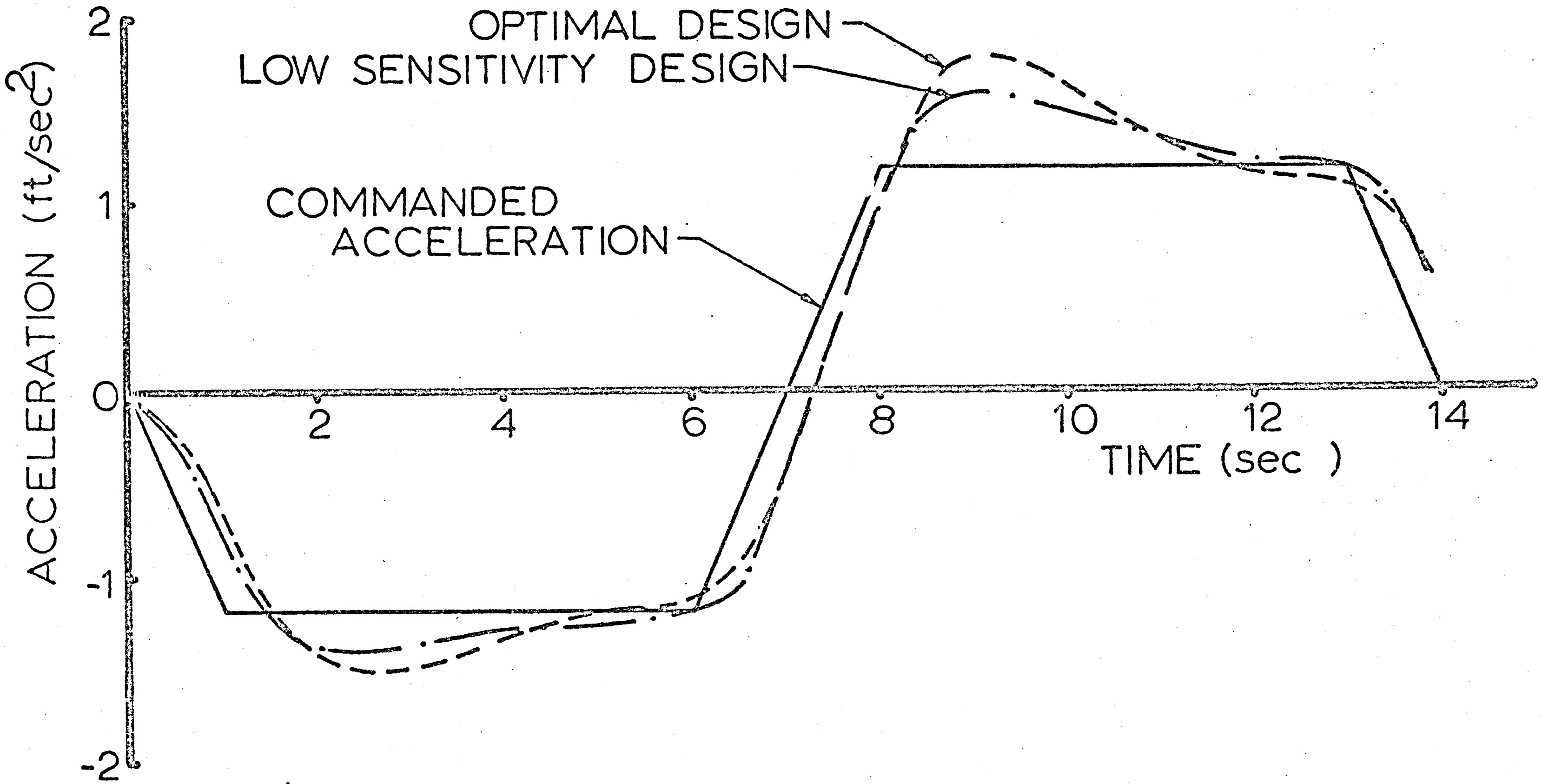


Fig.(5.26) Vehicle acceleration response during slot-slipping for various designs optimal design gain. Also, the maximum actual acceleration and jerk during merging and maneuvering are within the maximum allowed values (Section 5.2) as seen from Tables II, III.

	Merging from an off-line Station			
	Final Position Error (Ft)	Final Velocity Error (Ft/Sec)	Max. Acc. (Ft/ Sec^2)	Max. Jerk (Ft/ Sec^3)
Opt. Design	0.005	1.858	7.018	-7.720
Low Sen. Q	-0.058	1.839	6.901	-5.805
Low Sen. 10 Q	-0.395	1.754	6.538	-5.995

Table II Comparisons of system performance for various designs during merging from an off-line station

	Maneuvering or Slot Slipping			
	Final Position (Ft)	Final Velocity (Ft/Sec)	Max. Acc. (Ft/ Sec^2)	Max. Jerk (Ft/ Sec^3)
Opt. Design	-0.394	0.553	1.791	1.540
Low Sen. Q	-0.450	0.554	1.742	1.515
Low Sen. 10 Q	-0.658	0.506	1.596	1.435

Table III Comparisons of system performance for various designs during slot-slipping

Figures (5.27)-(5.28) show the velocity and position responses for the emergency stopping with a constant commanded deceleration of 10 ft/sec^2. The commanded stopping distance is 125 feet. The results for emergency stopping are summarized in Table IV.

	Optimal design gain	Low sen. Q_1	Low sen. $100Q_1$
Travelled distance (ft) at t=5 sec	129.918	130.064	130.503
Final velocity (ft/sec)	-1.628	-1.458	-0.950
Maximum acceleration (ft/sec^2)	-12.965	-12.712	-11.963
Maximum jerk (ft/sec^3)	-14.065	-14.875	-14.010

Table IV: Comparison of system performance for various designs during emergency stopping

Section 5.5 Simulation Results: Stochastic Case

The dynamic response of the longitudinal control system, designed in previous section for use, in the presence of input disturbance and sensor errors is studied in this section. The following situations were studied:

(1) The effects of eliminating the velocity sensor upon the system performance. (Section 5.5.2)

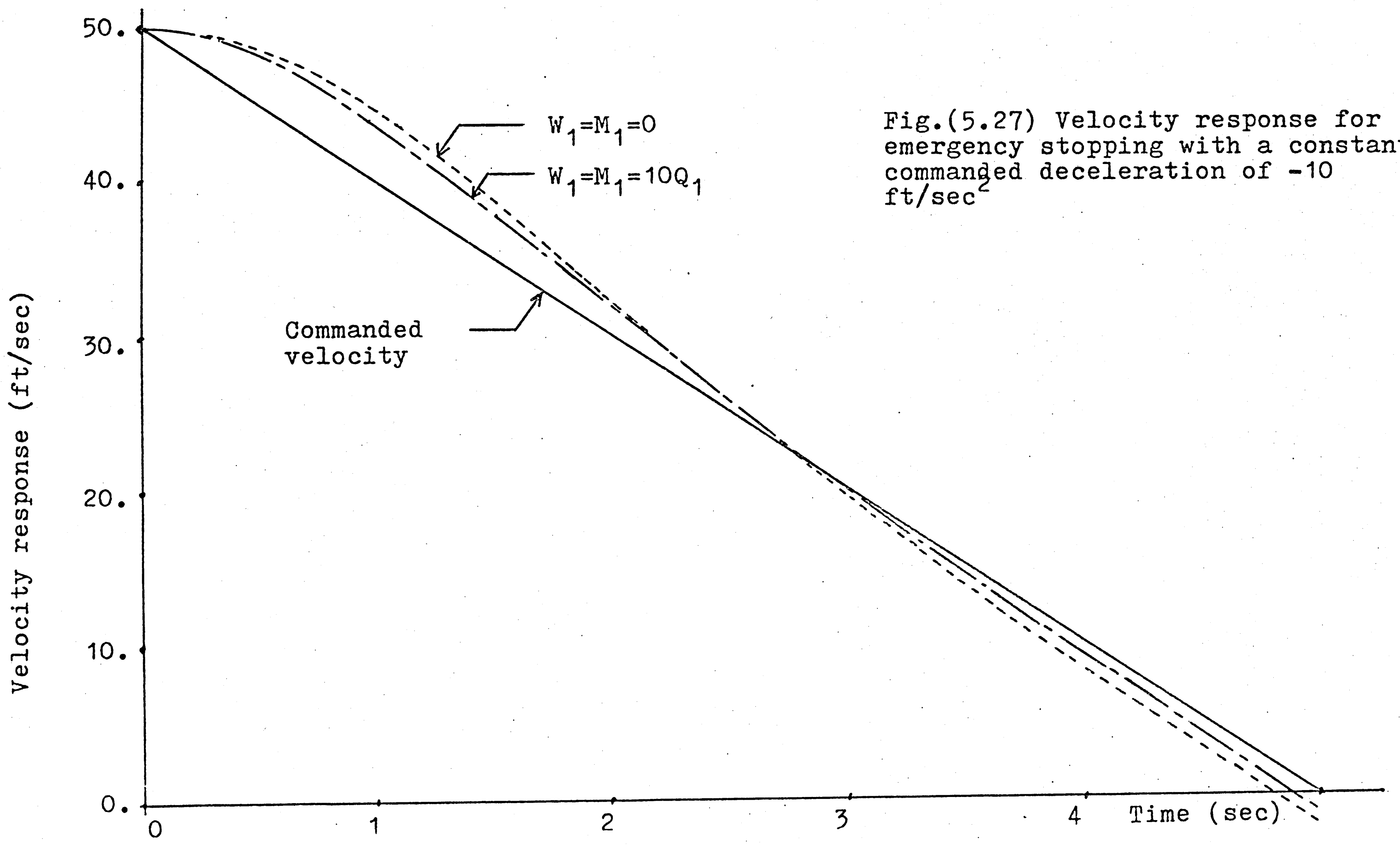


Fig.(5.27) Velocity response for emergency stopping with a constant commanded deceleration of -10 ft/sec^2

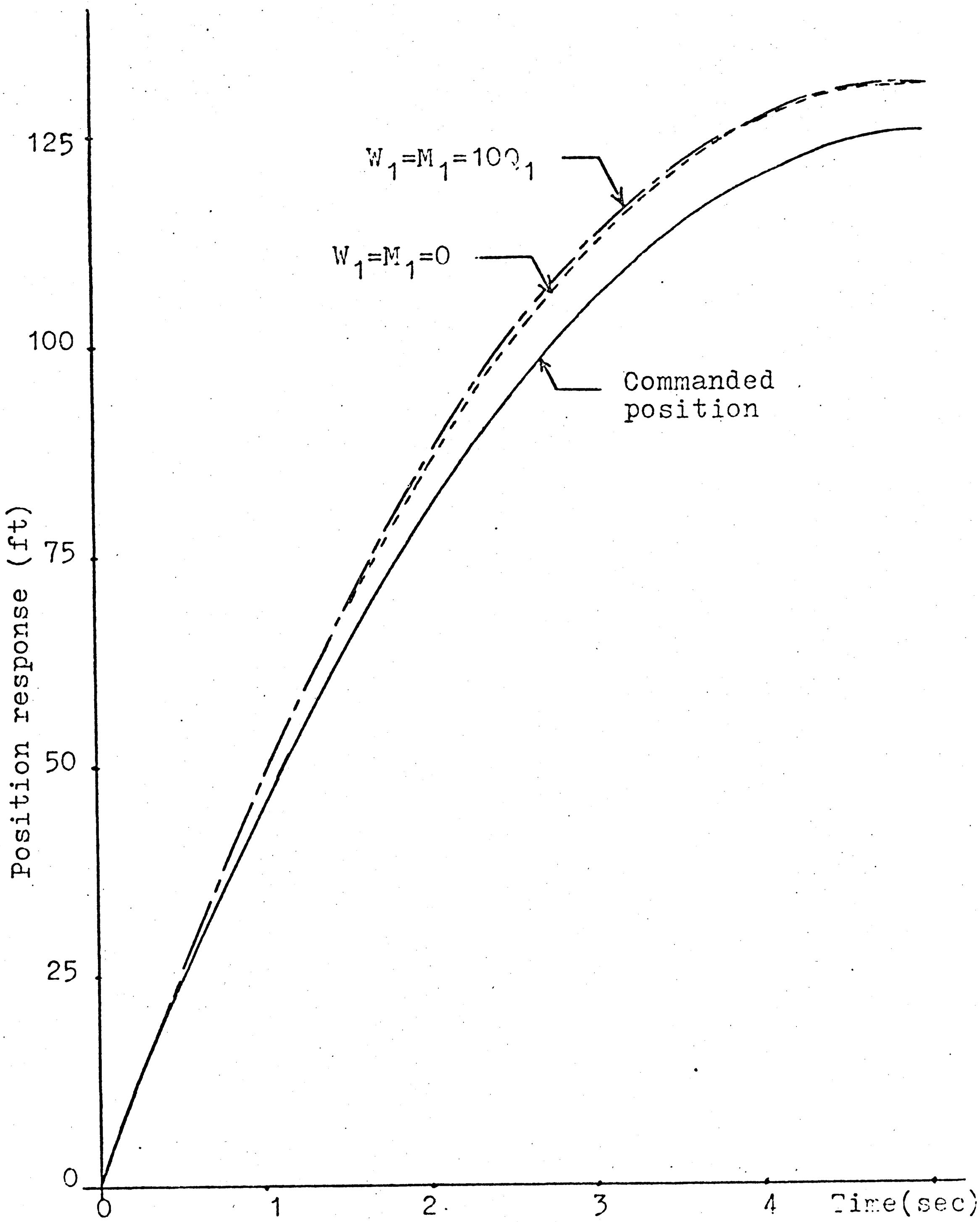


Fig.(5.28) Position response of emergency stopping with constant commanded deceleration of -10 ft/sec^2 for various designs

(2) Tradeoffs relating the steady-state RMS position, velocity, and acceleration errors to the accuracy of sensors for various designs. (Section 5.5.3)

(3) The effects of various disturbance environments upon the system performance. (Section 5.5.3)

(4) Comparisons between the control system using observers and the control system using estimators. (Section 5.5.4)

(5) The vehicle dynamic response of the estimators and the observers. (Section 5.5.4)

Section 5.5.1 Selection of Covariance Matrices Q_2 and R_2

In previous chapter, the input disturbance $d(t)$ and sensor error $v(t)$ are assumed to be white Gaussian processes with zero mean and constant covariance matrices Q_2 and R_2, respectively. The values of Q_2 and R_2 affect the filtering gain matrix H_2 which in turn affects the RMS errors of the dynamic response of the vehicle. One can choose a sensor having a given uncertainty R_2, but one cannot select the disturbance environment Q_2. In the following study, the performance results will be investigated as a function of sensor uncertainty.

For the system in which only position is measured, the covariance matrix R_2 is simply r_{11}. The sensor covariance matrix R_2 is diag.(r_{11}, r_{22}) for systems measuring position and velocity. The root of the covariance value, $\sqrt{r_{11}}$ (ft), is the uncertainty associated with the position measurement and $\sqrt{r_{22}}$ (ft/sec) is the uncertainty associated with the velocity measurement. Since the velocity measurement can be done easily with more accuracy, $\sqrt{r_{22}}$ is assumed to be $\frac{1}{2}\sqrt{r_{11}}$. In our study the values of the position uncertainty were chosen to be $\frac{1}{16}$ ft, $\frac{1}{8}$ ft, $\frac{1}{4}$ ft, $\frac{1}{2}$ ft, 1 ft, and 2 ft. The performance results which were expressed in terms of the steady-state RMS values were obtained as a function of the sensor uncertainty $\sqrt{r_{11}}$.

The input disturbance $d(t)$ is assumed to be a white Gaussian process with zero mean and constant covariance matrix Q_2, which is diag.$(0, 0, q_{33}, q_{44})$ for the control system under study. In the following the input disturbance model due to wind gusts were made by using the wind gust data collected in Los Angeles and shown in Table V. (see [S4])

	Typical day (slight breeze)	Extremely gusty day
RMS wind gust (ft/sec)	5	30
Max. wind speed (ft/sec)	10	70 to 90
Correlation time (sec)	300	30

Table V Wind gust data for Los Angeles

From equation (2.12), the disturbance force due to wind gust is obtained as

$$\dot{f}_d = \frac{2C_aH}{M}(1+w)\dot{w} \tag{5.5}$$

where w is the nondimensional wind gust velocity. The input disturbance due to wind gust, $d_{33}(t)$, is modeled as a first order random process described by ([S7] , [B2])

$$\dot{V}_w = \frac{V_w}{\mathcal{T}} + N(t) \tag{5.6}$$

where $\mathcal{T}$ is the correlation time of the wind gust

N is a white Gaussian process with zero mean and constant covariance q_w, i.e.,

$$E\left[N(t)\right] = 0 \tag{5.7}$$

$$\text{cov.}\left[N(t);N(\tau)\right] = q_w(t-\tau) \tag{5.8}$$

The covariance q_w of the wind gust is related to the RMS magnitude of the wind gust by ([B2])

$$q_w = 2\mathcal{T}(RMS)^2 \tag{5.9}$$

Nondimensionalizing equation (5.6) and combining with (5.5) gives

$$\dot{f}_d = \frac{2C_aH}{M}(1+w)\left(\frac{Tw}{\mathcal{T}} + \frac{TN}{V_c}\right) \tag{5.10}$$

Using the data in Table V, the values of the external disturbance due to the wind gusts were calculated as*

*The number behind E means an exponent representing a power of 10. For example, 2.0E-3 means 2.0×10^{-3}.

$$\dot{f}_d = \begin{cases} 1.33E{-}5+(1.63E{-}4)N(t) & \text{for light breeze day} \\ 2.8\ E{-}3+(7.23E{-}3)N(t) & \text{for extremely gusty day} \end{cases} \tag{5.11}$$

The corresponding values of the covariance matrix Q_2 are

$$q_{33} = \begin{cases} 2.66E{-}8 & \text{for light breeze day} \\ 5.22E{-}5 & \text{for extremely gusty day} \end{cases} \tag{5.12}$$

In our study, the values of q_{33} were chosen to be 4.0E-8 (for light breeze day) and 6.4E-5 (for extremely gusty day). An intermediate case in which q_{33}= 1.6E-6 will also be studied. This corresponds to the gusty day when RMS wind gust speed is 20 ft/sec with maximum wind gust speed up to 50 ft/sec and the correlation time 90 seconds. The values of q_{44} were assumed to be of $\frac{1}{10}q_{33}$. The effects of the various external disturbances upon the control system performance are given in Section 5.5.3.

Section 5.5.2 The Effects of Eliminating the Velocity Sensor

The sensor was modeled by (Section 4.2.1)

$$z = Cx + v \tag{5.13}$$

where $C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$ if only position is measured

$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ if both position and velocity are measured.

From equations (4.66)-(4.67), the filtering gain matrices H_2 with and without the velocity sensor are obtained and given in Tables VI and VII. The flow charts of computer programs are listed in Appendix C. Using these filtering matrices we obtained the steady-state average behavior of the control system using the optimal estimator (Kalman-Bucy filter (4.65)-(4.69)) for the low sensitivity design $W_1=M_1=Q_1$. These are summarized in Table VIII.
From Table VIII, it is seen that the control system without the velocity measurement will experience the steady-state RMS position and velocity errors 40% larger than the control system with the velocity measurement. The use of the velocity sensor will decrease not only the RMS values of position and velocity errors, but also the RMS values of acceleration and jerk errors which are not shown in Table VIII. Thus, the control system is assumed to have the velocity sensor in the measuring process in the next sections.

Section 5.5.3 The Steady-state Performance Results

Figures (5.29)-(5.32) shows the steady-state RMS errors of the system performance as a function of sensor uncertainty for various designs. The RMS errors consist of two parts; one is the error due to the estimated state and the other is the estimation error which is independent of the design chosen. It is

	Sensor uncertainty $\sqrt{r_{11}}$ (ft)					
	1/16		1/8		1/4	
Q_2=diag.(0, 0,1.6E-6, 1.6E-7)	4.984E-1	9.208E-1	4.956E-1	8.674E-1	4.889E-1	7.905E-1
	2.302E-1	1.939	2.169E-1	1.332	1.976E-1	8.963E-1
	7.633E-2	1.986	6.339E-2	9.810E-1	4.968E-2	4.798E-1
	3.178E-7	1.684E-4	1.410E-7	4.471E-5	5.827E-8	1.167E-5

	Sensor uncertainty $\sqrt{r_{11}}$ (ft)					
	1/2		1		2	
Q_2=diag.(0, 0,1.6E-6, 1.6E-7)	4.752E-1	6.893E-1	4.515E-1	5.699E-1	4.158E-1	4.447E-1
	1.723E-1	5.853E-1	1.425E-1	3.673E-1	1.112E-1	2.199E-1
	3.642E-2	2.307E-1	2.481E-2	1.080E-1	1.569E-2	4.891E-2
	2.232E-8	3.010E-6	7.914E-9	7.691E-7	2.601E-9	1.952E-7

Table VI Filtering gain matrix H_2 for various sensor uncertainties when measuring position and velocity; $R_2=\text{diag.}(r_{11}, \frac{1}{4}r_{11})$
$H_2=K_2C^TR_2^{-1}$

	Sensor uncertainty $\sqrt{r_{11}}$ (ft)					
	1/16	1/8	1/4	1/2	1	2
Q_2=diag.(0, 0, 1.6E-6, 1.6E-7)	1.991	1.577	1.248	9.874E-1	7.803E-1	6.160E-1
	1.983	1.244	7.792E-1	4.875E-1	3.044E-1	1.897E-1
	9.789E-1	4.852E-1	2.399E-1	1.183E-1	5.813E-2	2.843E-2
	4.190E-6	1.092E-6	2.820E-7	7.236E-8	1.847E-8	4.693E-9

Table VII Filtering gain matrix H_2 for various sensor uncertainties when measruing position only; $R_2 = r_{11}$, $H_2=K_2C^TR_2^{-1}$

RMS sensor uncertainty $\sqrt{r_{11}}$ (ft)	RMS position error (ft)		RMS position estimation error as % of total position error		RMS velocity error (ft/sec)		RMS velocity estimation error as % of total velocity error	
	with vel sensor	without vel sensor	with vel sensor	without vel sensor	with vel sensor	without vel sensor	with vel sensor	without vel sensor
1/16	.289	.583	2.34%	2.28%	.109	.205	16.0%	27.6%
1/8	.415	.789	4.50	3.98	.149	.266	23.49	32.70
1/4	.633	1.097	7.62	6.48	.212	.351	31.02	37.29
1/2	1.009	1.569	11.55	10.03	.313	.471	37.37	40.96
1	1.650	2.303	16.57	14.71	.469	.643	41.76	43.45
2	2.731	3.466	22.29	20.52	.707	.891	44.02	44.58

Table VIII Steady-state RMS results for low sensitivity design $W_1=M_1=Q_1$, with and without velocity sensor; Q_1=diag.(10,100,10,10) Q_2=diag.(0,0,1.6E-6,1.6E-7) observed in these figures that the low sensitivity design gives better RMS values of velocity, acceleration, and jerk at the expense of the RMS position error. However, the variations of the RMS values due to different designs are small. The RMS errors increase as the sensor uncertainty increases for a given disturbance environment Q_2. If the acceptable level of RMS errors of the control system is given, Figs. (5.29)-(5.32) could be used to determine the necessary sensor uncertainty. The proposed procedures for doing this will be discussed below.

Figs.(5.33)-(5.36) show the steady-state RMS errors of the system performance for low sensitivity design $W_1=M_1=Q_1$ under various disturbance environments Q_2. The RMS values increase as the disturbances increase. Figures (5.29)-(5.36) should be useful in the design of the feedback longitudinal control system for automated transit vehicles. For example, suppose that the worst RMS value of the disturbance environment is known to have the nondimensional value of 6.4E-5 and suppose that the maximum tolerance for the RMS position error is 1.5 ft and the maximum tolerance for the RMS velocity error is 0.75 ft/sec, then the sensor uncertainty must be designed to be less than $\frac{1}{4}$ ft. The use of the more accurate sensor will increase the cost of the measurement system.

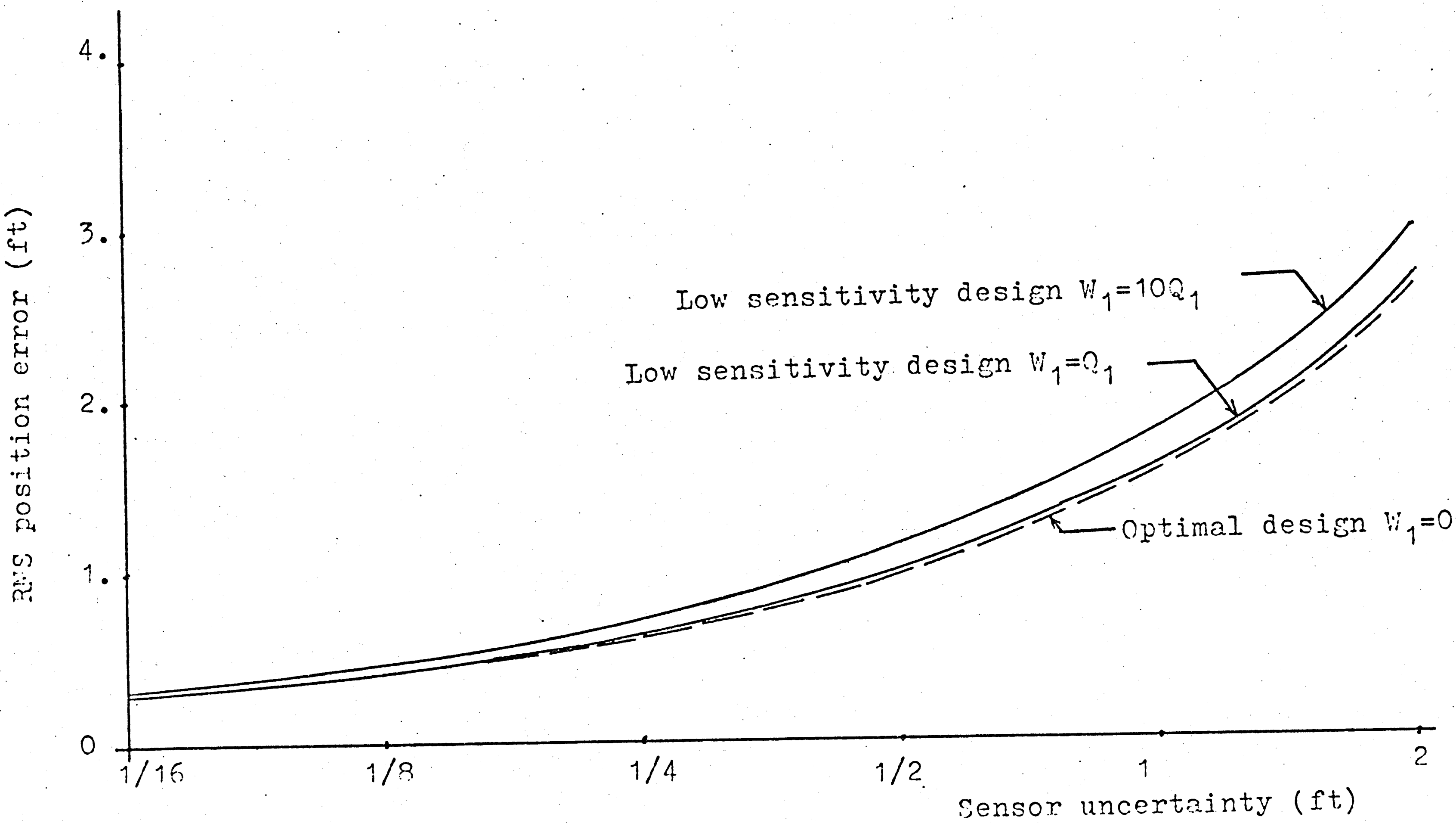


Fig.(5.29) Steady-state RMS position error vs. sensor uncertainty for various designs; Q_1=diag.(10,100,10,10), Q_2=diag.(0,0, 1.6E-6, 1.6E-7)

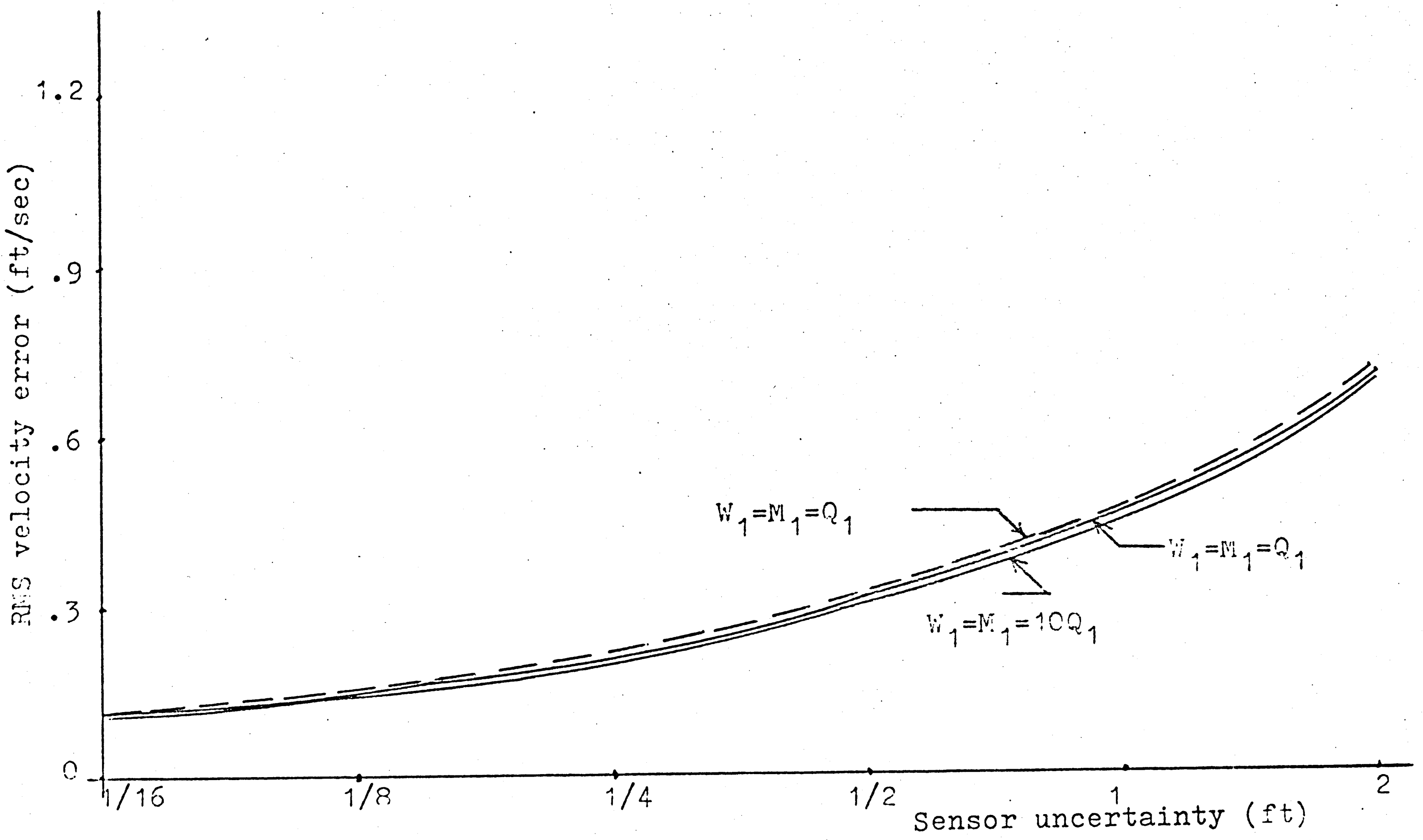


Fig.(5.30) Steady-state RMS velocity error vs. sensor uncertainty for various design; Q_1=diag.(10,100,10,10), Q_2=diag.(0,0, 1.6E-6, 1.6E-7)

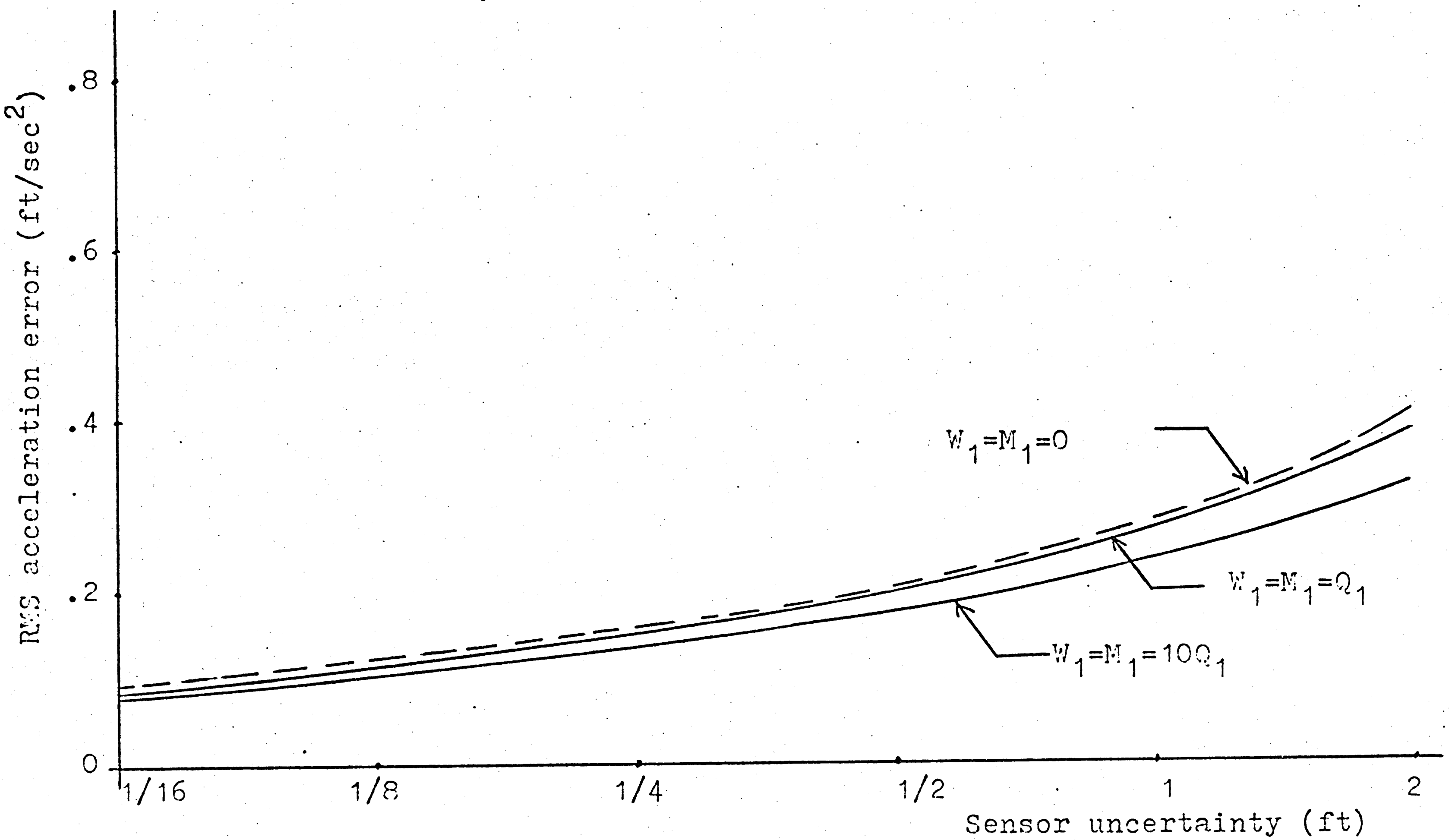


Fig.(5.31) Steady-state RMS acceleration error vs. sensor uncertainty for various design; Q_1=diag.(10, 100, 10, 10) Q_2=diag.(0, 0, 1.6E-6, 1.6E-7)

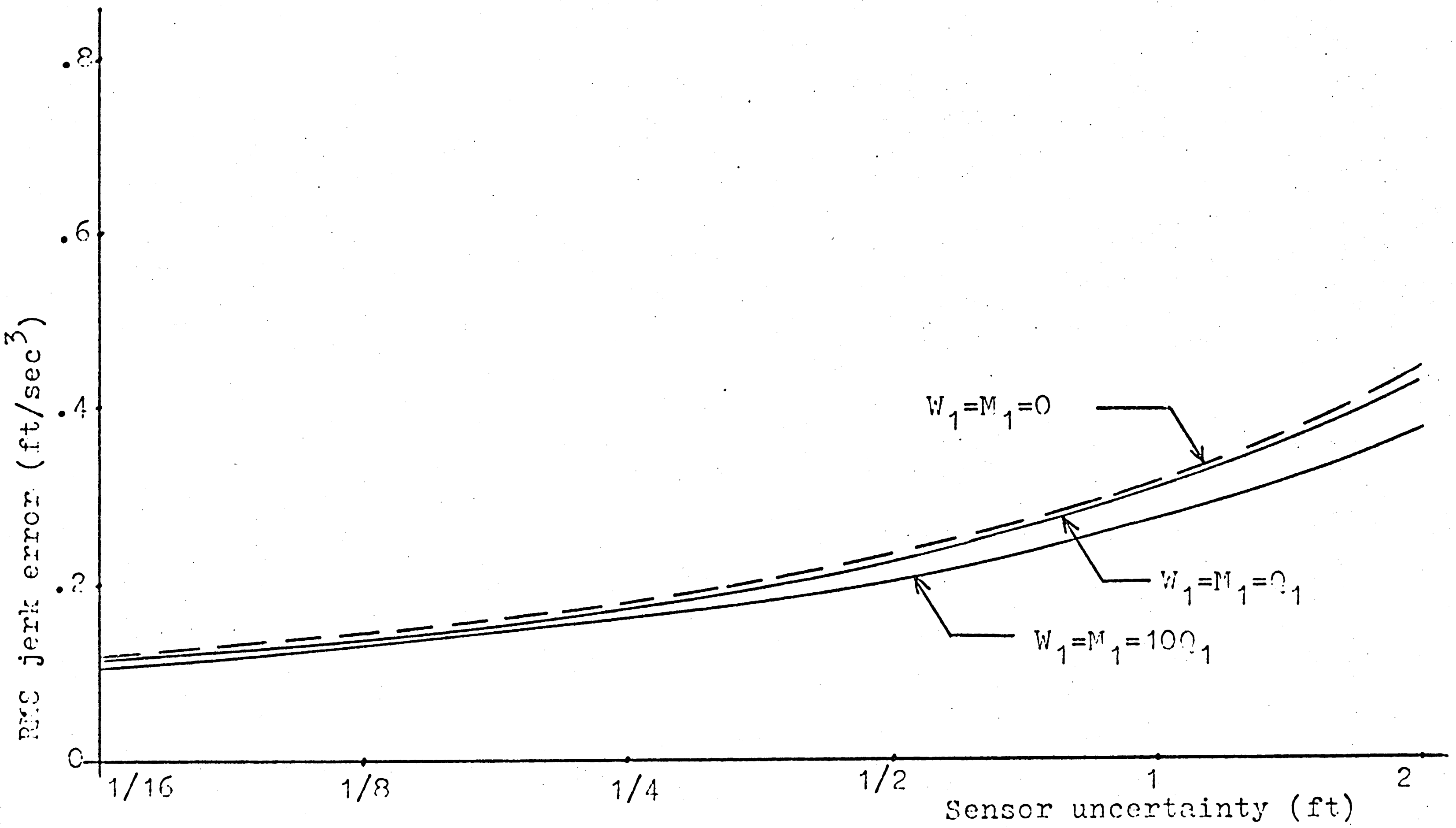


Fig.(5.32) Steady-state RMS jerk error vs. sensor uncertainty for various design; Q_1=diag.(10,100,10,10) Q_2=diag.(0,0,1.6E-6, 1.6E-7)

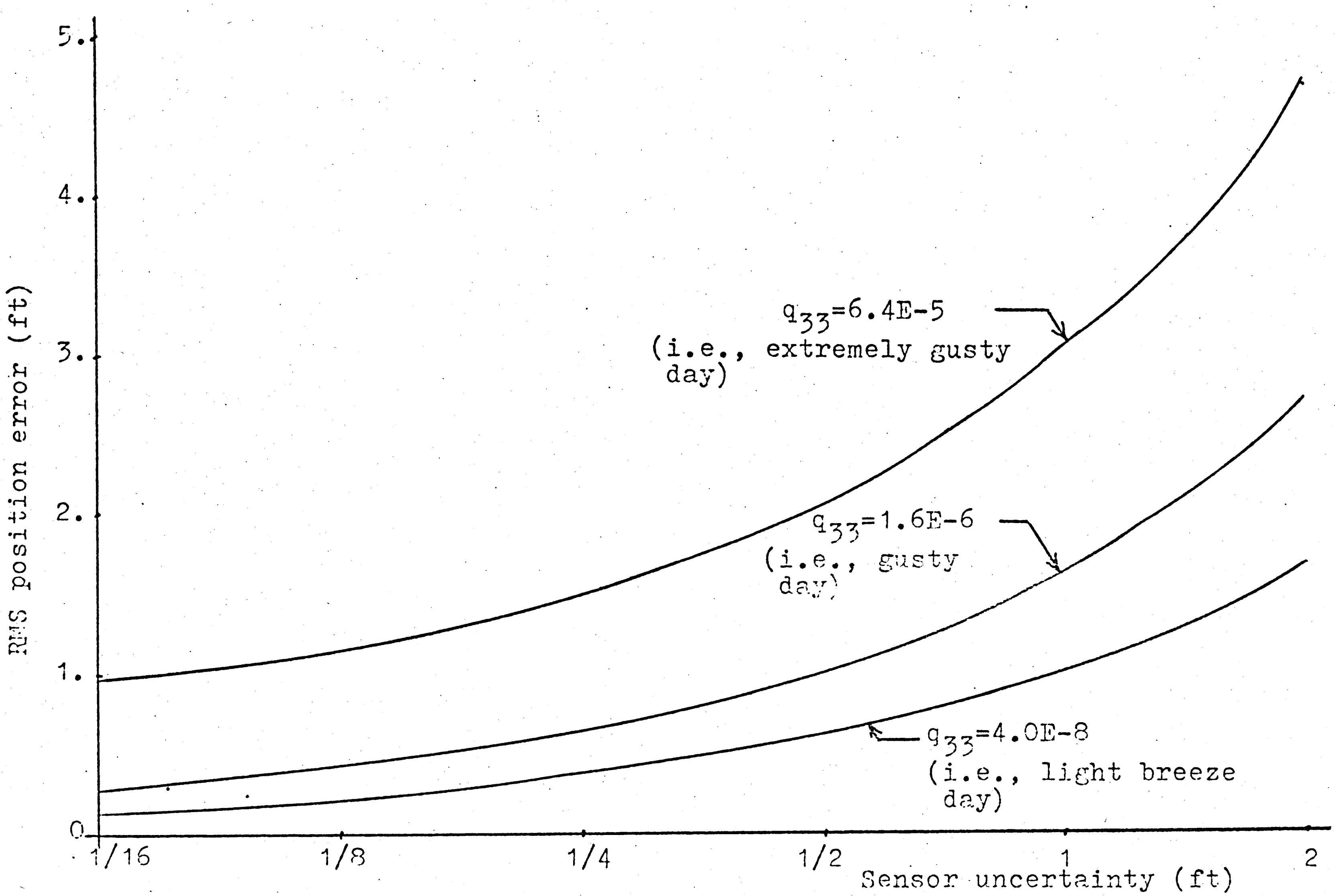


Fig.(5.33) Steady-state RMS position error vs. sensor uncertainty for various disturbance environments using the low sensitivity design $W_1=M_1=Q_1$ and the optimal estimator (4.65)-(4.69)

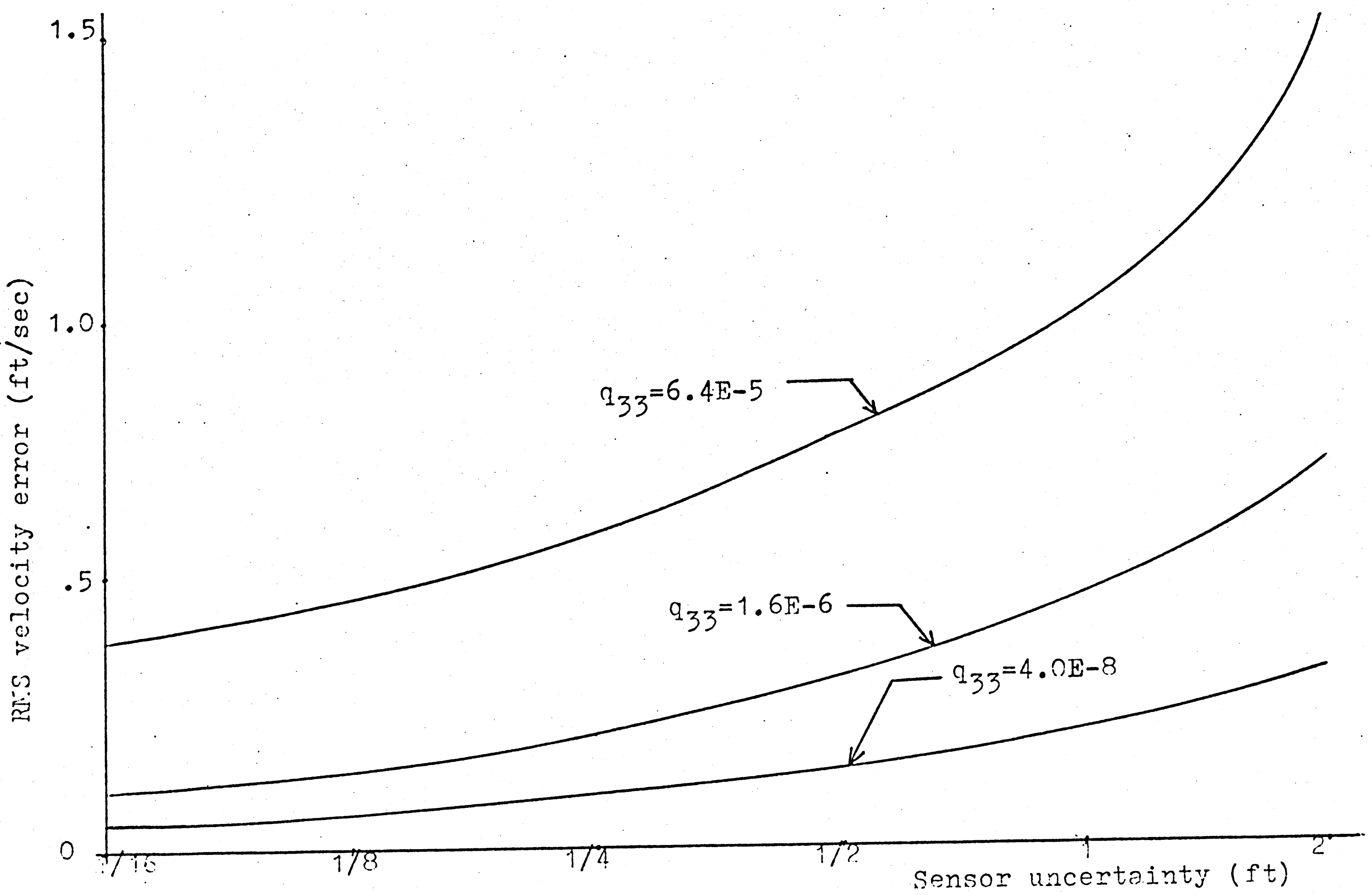


Fig.(5.34) Steady-State RMS velocity error vs. sensor uncertainty for various disturbance environments .

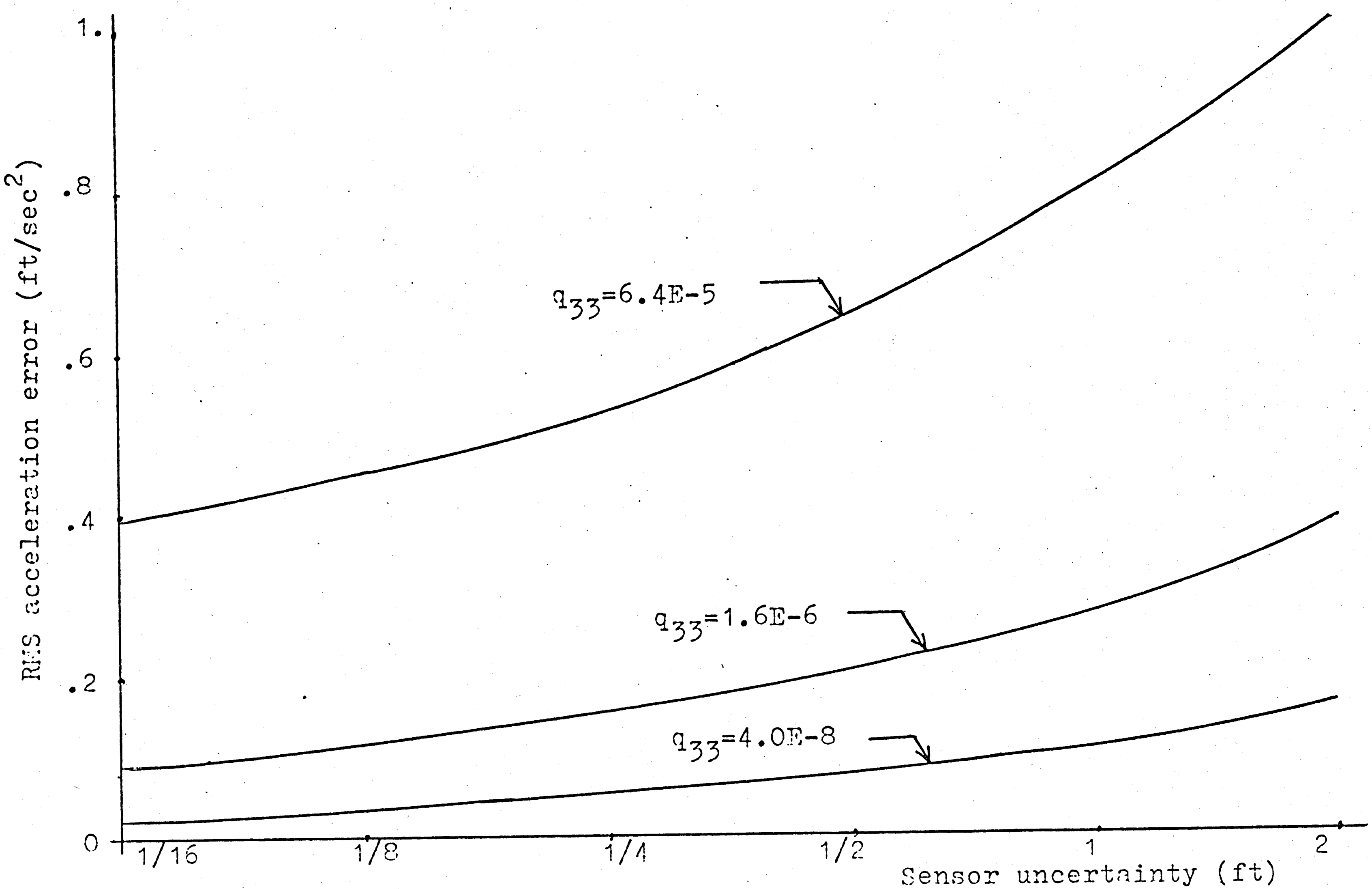


Fig.(5.35) Steady-state RMS acceleration error vs. sensor uncertainty for various disturbance environments.

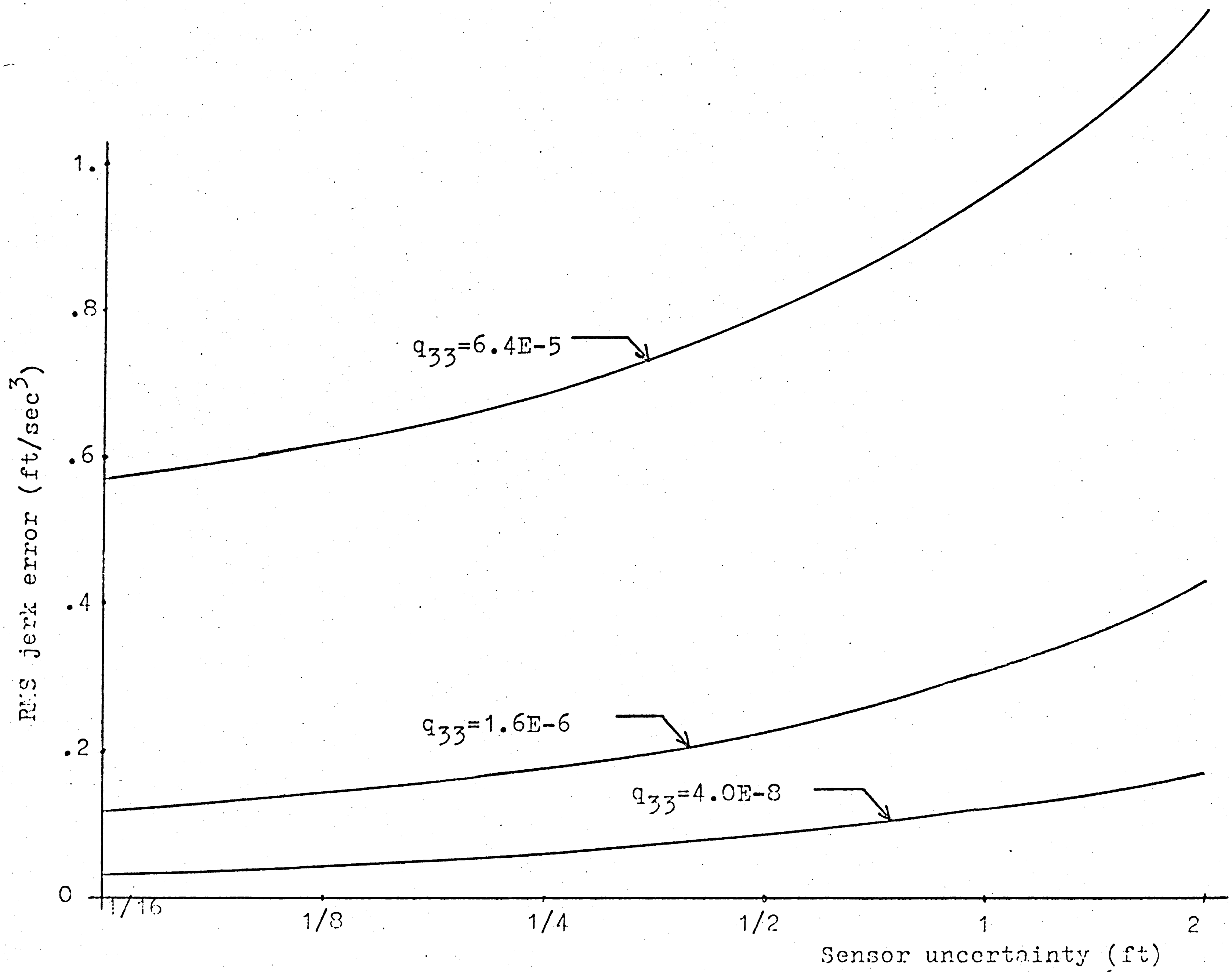


Fig.(5.36) Steady-state RMS jerk error vs. sensor uncertainty for various disturbance environments.

Section 5.5.4 Comparisons Between Estimators and Observers

An observer is an alternate to the estimator for obtaining the missing state variables of the control system. ([L7], [G4]). The advantage of using observer is that its dimension is smaller than the dimension of the system using the Kalman-Bucy filter. In this section, comparisons were made between the control system using the optimal estimator and the control system using the observer.

The steady-state behavior of the control system using an observer in the presence of input disturbances and sensor errors are derived and given in Appendix D. The covariance matrix of the estimation error for observers consists of two terms as shown in equation (D.16), i.e.,

$$K_4 \triangleq E\left[(x-\hat{x})(x-\hat{x})^T\right] = P_1R_2P_1' + P_2K_3P_2' \qquad (5.14)$$

where R_2 is the covariance matrix of sensor errors and K_3 is the covariance matrix of the estimated state variables satisfying the following equation

$$0 = DK_3 + K_3D' + FR_2F' - TQ_2T' \qquad (5.15)$$

The matrices P_1 and P_2 depend on the choice of the observer matrices D, F, and T in equations (D.3)-(D.7). We will use the same value of observer matrices as those used in Ref. [G4]. That is, the observer matrices D, F, and T are chosen to be

$$D = \begin{bmatrix} -12 & -1 \\ 0 & -12 \end{bmatrix} \qquad F = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \tag{5.16}$$

$$T = \begin{bmatrix} 0 & t_{12} & t_{13} & t_{14} \\ 0 & t_{22} & t_{23} & t_{24} \end{bmatrix}$$

The resulting steady-state RMS errors for position and velocity measurements are the same as sensor measurement errors. The acceleration and jerk errors for observers and estimators are as shown in Fig. (5.37). The overall root-mean-square error in the estimation of the state variables is shown in Fig. (5.38).

The stochastic responses of the vehicle in reducing an initial position error of 2.5 feet for gusty days (q_{33}= 1.6E-6) with $\frac{1}{8}$ ft sensor uncertainty are shown in Figs.(5.39)-(5.42). There are three trajectories shown in each figure, i.e., the actual response, the response obtained by using the optimal estimators (Kalman-Bucy filter) and the response obtained by using the observer. They will be called the actual, estimated, and observed responses respectively. Fig.(5.39) shows good estimated position response which is closer to the actual response than the observed one. Fig.(5.40) compares the actual, estimated,

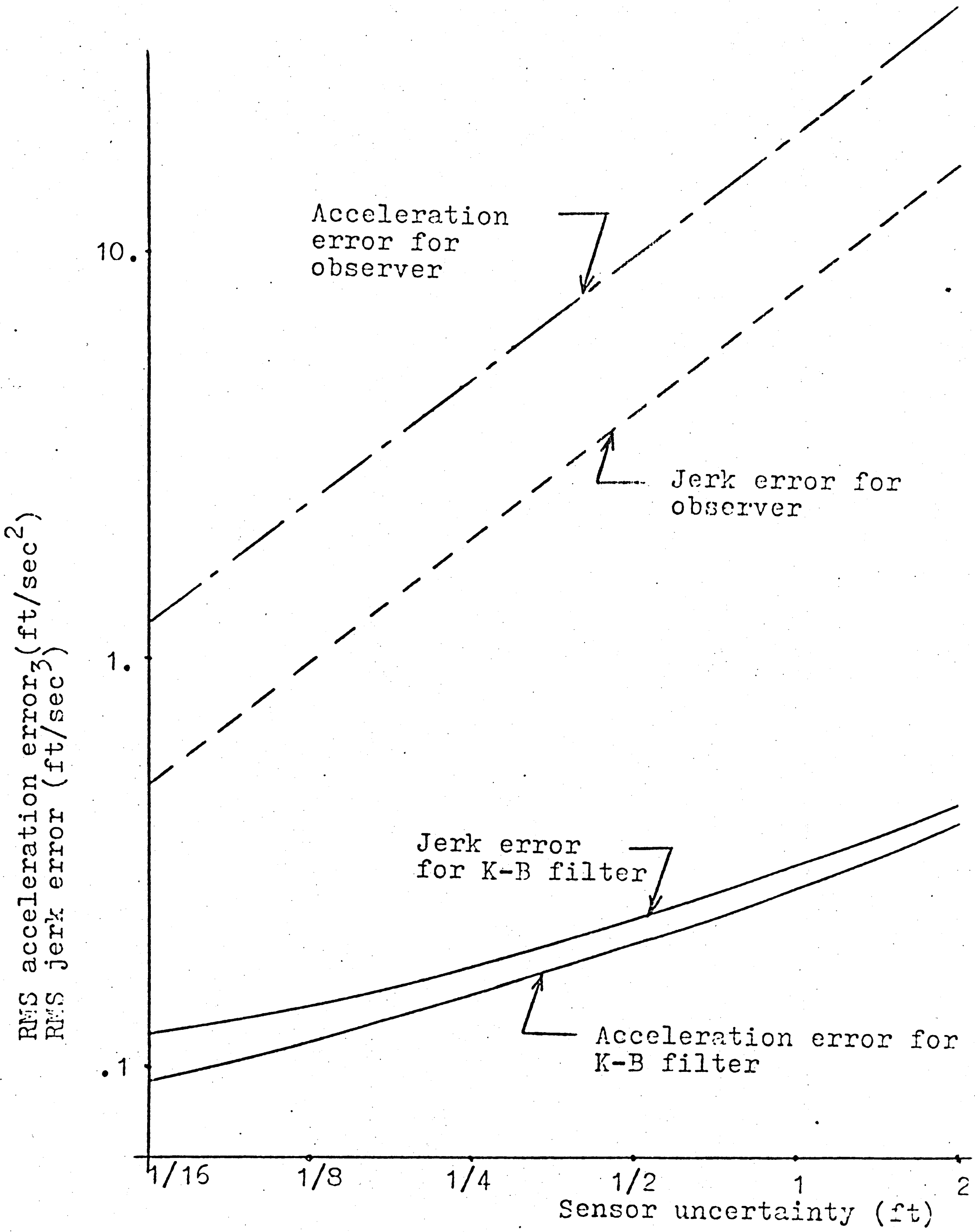


Fig.(5.37) The acceleration and jerk errors of observer and Kalman-Bucy filter for low sensitivity design $W_1=N_1=Q_1$; Q_2=diag.(0,0,1.6E-6, 1.6E-7)

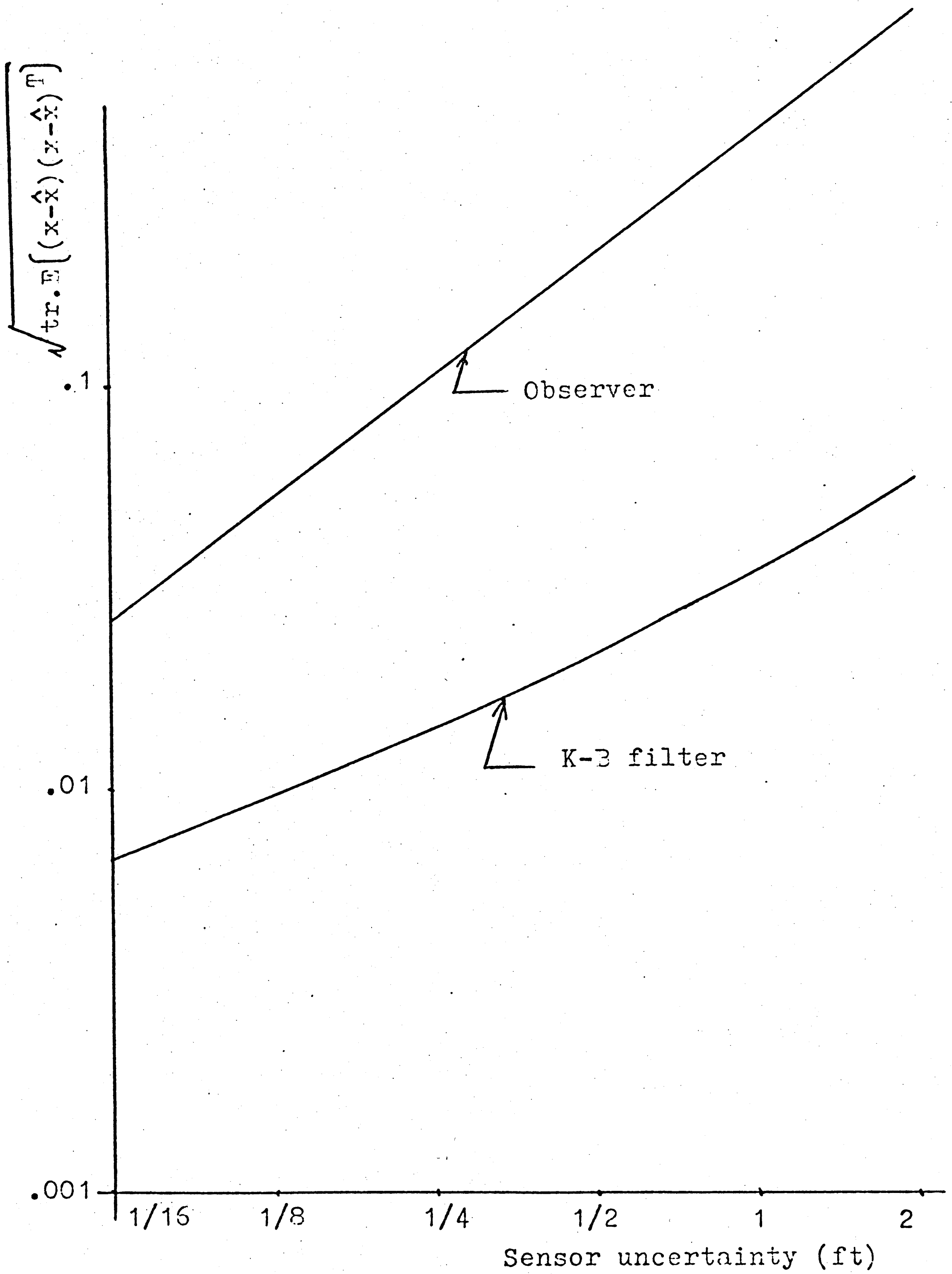


Fig.(5.38) The overall nondimensional RMS estimation error of observer and Kalman-Bucy filter for low sensitivity design $W_1=M_t=Q_1$; Q_2=diag.(0,0,1.6E-6, 1.6E-7)

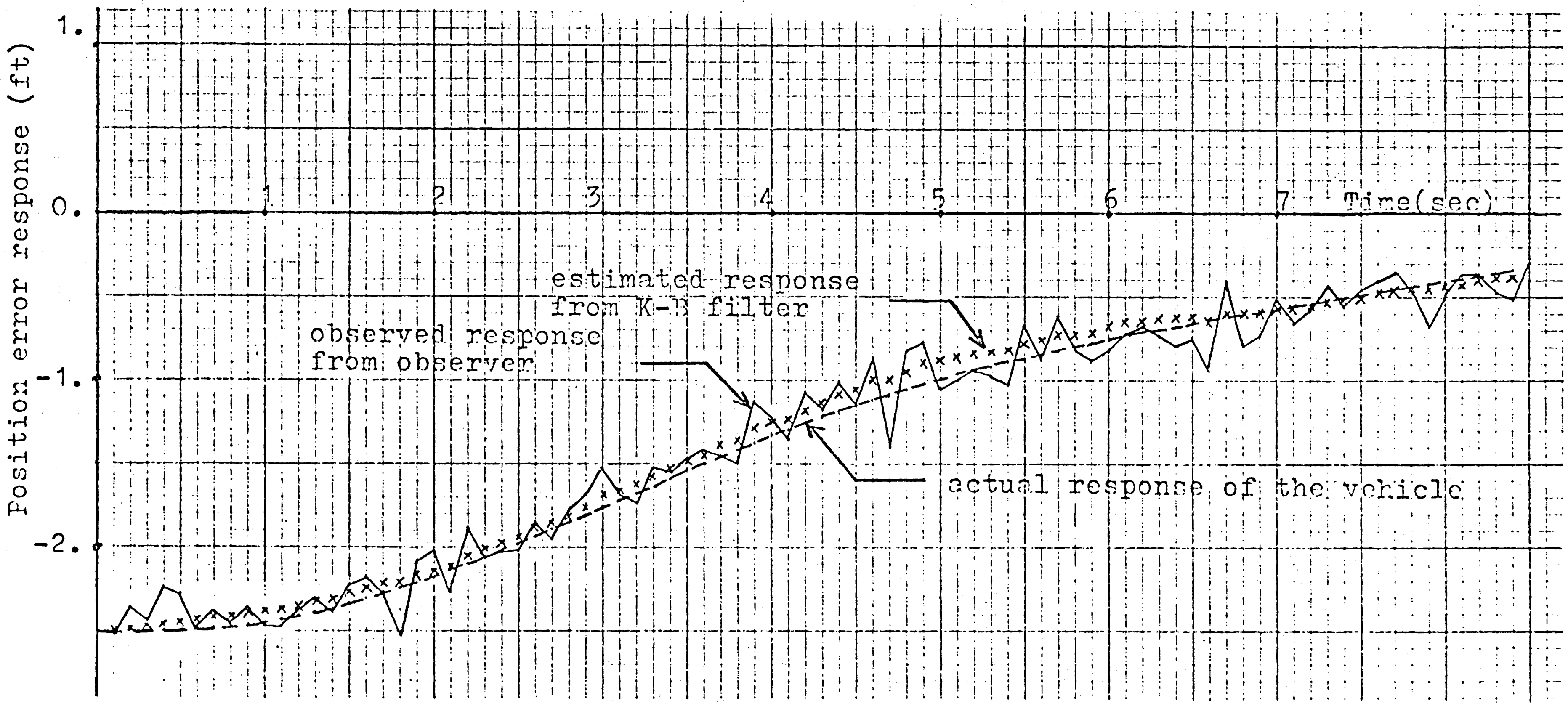


Fig.(5.39) Stochastic position error response in reducing an initial position error of 2.5 feet by using low sensitivity design $W_1=M_1=Q_1$; $Q_1=$ diag.(10,100,10,10), $Q_2=$diag.(0,0,1.6E-6, 1.6E-7), $R_2=$diag.(6.25E-6, 1.5625E-6)

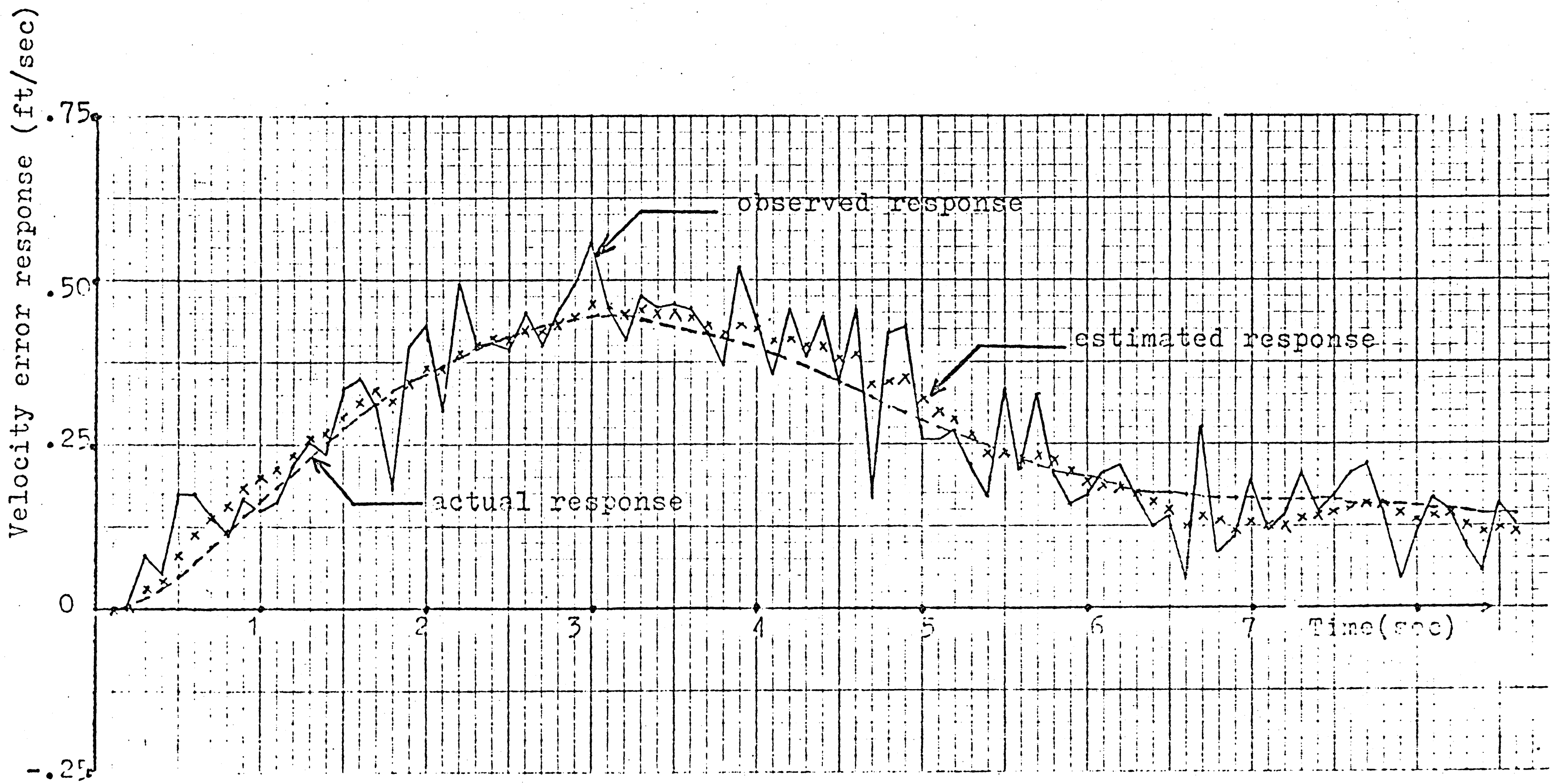


Fig.(5.40) Stochastic velocity error response in reducing an initial position error of 2.5 feet by using low sensitivity design Q_1.

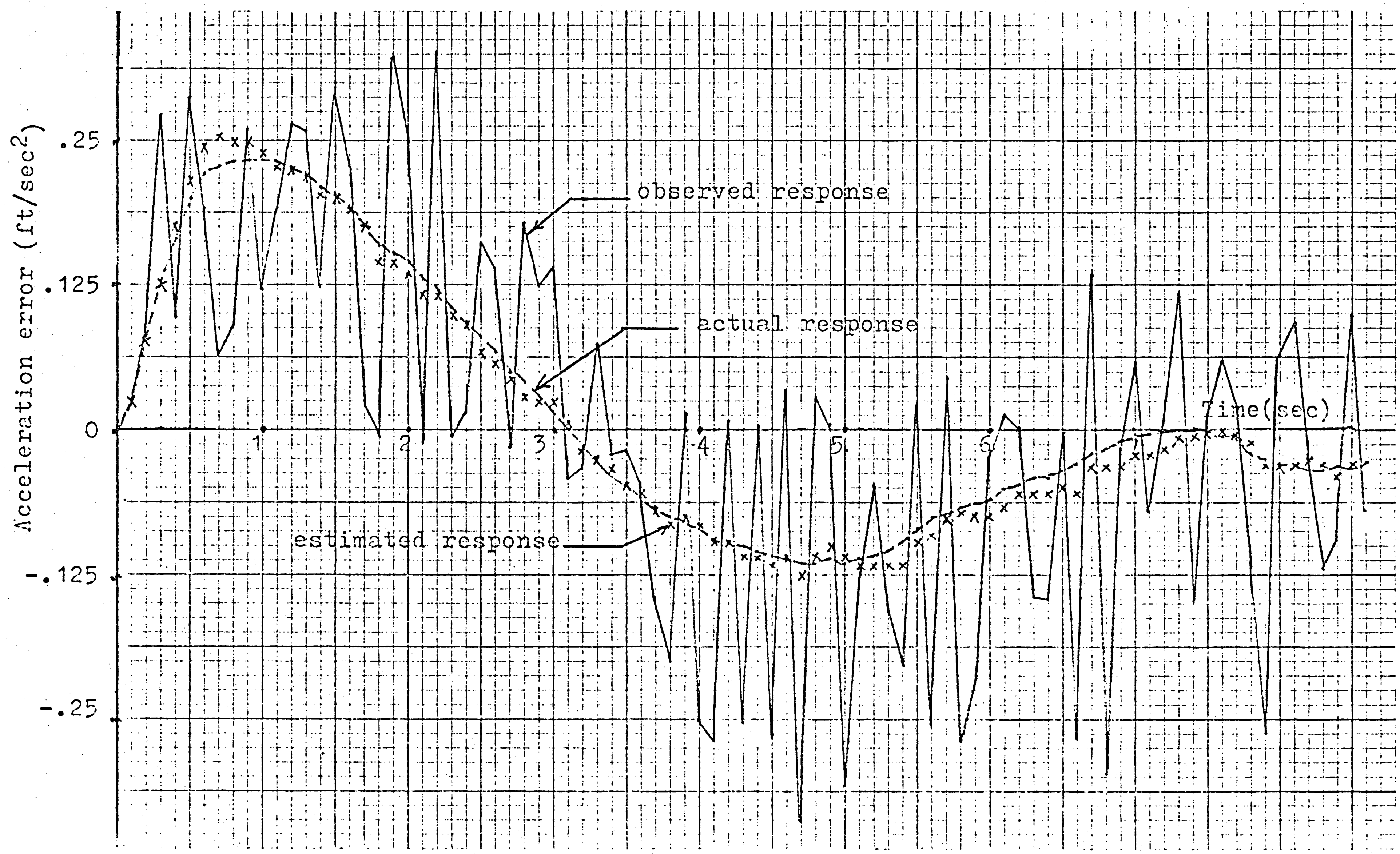


Fig.(5.41) Stochastic acceleration error response in reducing an initial position error of 2.5 feet by using low sensitivity design G_1

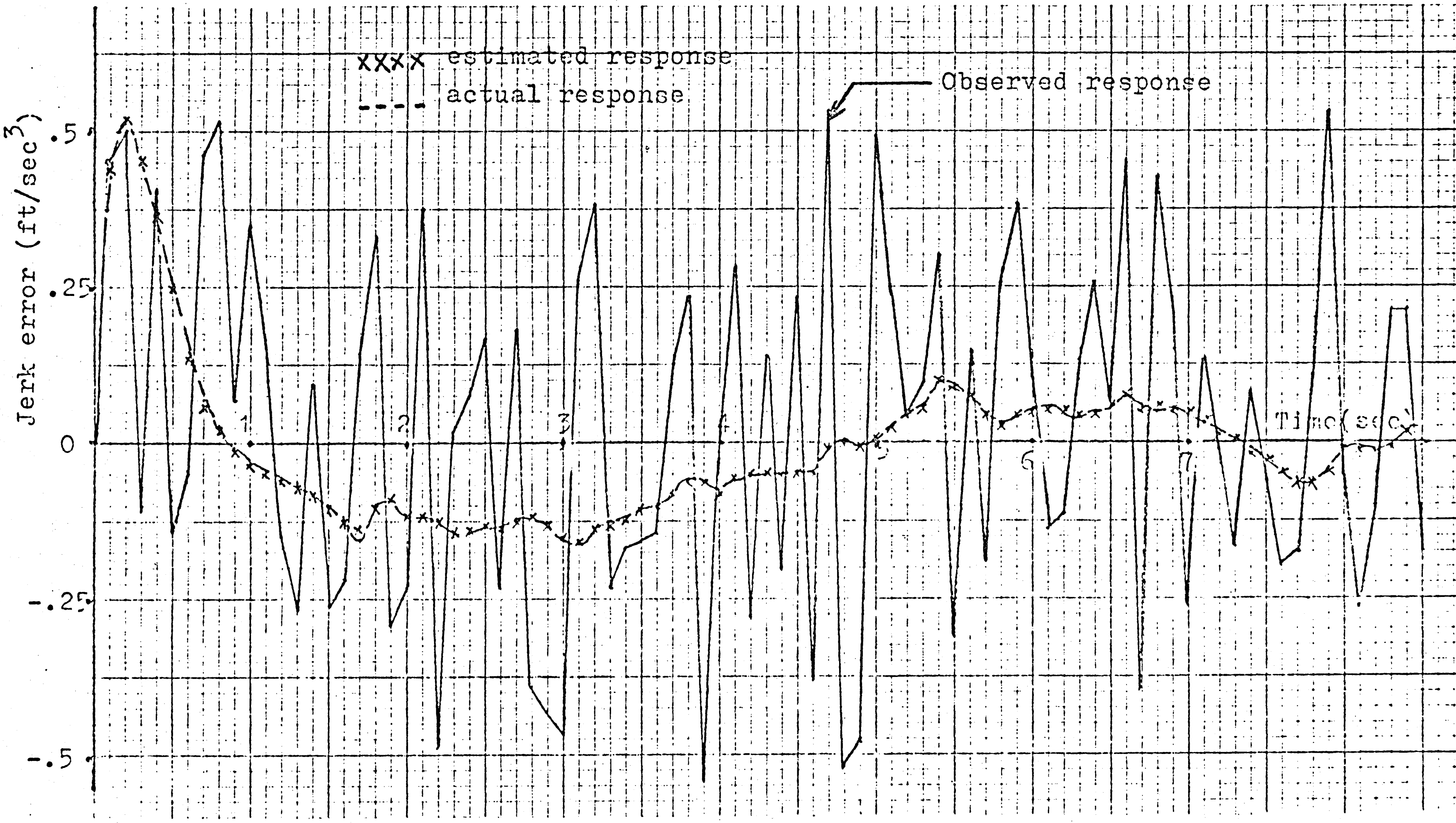


Fig.(5.42) Stochastic jerk error response in reducing an initial position error of 2.5 feet by using low sensitivity design Q_1

and observed velocity responses of the vehicle. These two figures show that the optimal estimator filters the noisy measurement of the position and velocity responses. The observed acceleration response shows poor match to the actual acceleration response as seen from Fig.(5.41). This is because the large estimation error in the total acceleration error in the observer system. [Fig.(5.37)] . The same situation happened in the jerk response Fig.(5.42). Therefore, an optimal estimator is shown not only to smooth the measured state variables, but also to provide an estimate of the unmeasured state variables which are better than those obtained by using the observer.

Section 5.6 Summary of Example

In this chapter, the design of a longitudinal control system for a high-capacity PRT system is studied in detail. Typical vehicle data and control system specifications are given in Section 5.2. The method used to determine the control gain is investigated in Section 5.3. The simulation results for normal mainline operation and nonlinear profile-following operation are presented in Section 5.4. The dynamic responses show that the feedback control system gives tight position control as well as a comfortable ride. In the presence of stochastic disturbances and sensor errors, it is found that the use of the velocity sensor will reduce the steady-state RMS errors as shown in Section 5.5.2. In Section 5.5.3, a procedure for determining the maximum allowable sensor uncertainty is proposed. The control system using estimators is found insensitive to the parameter variations. The observer of lower dimensionality than the optimal estimator (Kalman-Bucy type filter) is shown to be less suitable for use in the presence of the stochastic disturbances and sensor errors than the optimal estimator. (Section 5.5.4). The optimal estimator is shown to smooth the measured state variables, and to provide an estimate of the unmeasured state variables which are necessary in our design. (Section 5.5.4)

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Section 6.1 Conclusions and Contributions of the Thesis

From the results shown in Chapter 5, it appears that optimal control theory can be usefully applied to the design of longitudinal control system for automated transit vehicles. The resulting control systems keep headway and velocity errors small without causing passengers discomfort and excellent dynamic response is achieved during mainline operation as well as during nonlinear profile-following operation. The use of low sensitivity design results in a smooth ride at the expense of a longer time required to reduce position error. The controllers in both designs are linear with constant gains and should be relatively economical to implement. In the presence of stochastic disturbances and imperfect sensing, the use of velocity sensor is recommended. The optimal estimator (Kalman-Bucy filter) is shown to give better results than the observer in the presence of sensor errors.

The contributions of the thesis can be summarized as follows:

(1) New mathematical modelling of automated transit vehicles in an external reference system, including the rolling resistance and propulsion system dynamics.

(2) The development of a new theory of low sensitivity design with state feedback control structure for a class of linear time-invariant systems.

(3) The development of a new theory of low sensitivity design in the presence of stochastic disturbances and sensor errors for a class of linear stochastic time-invariant systems.

(4) The use of above theories to design a vehicular longitudinal control system for a specific PRT system and to test by computer simulation the applicability of the resulting longitudinal controllers, observers, and optimal estimators.

Section 6.2 Suggestions for Future Research

In the study of the longitudinal control of automated transit vehicles in this thesis, all equations are modeled as continuous systems. A related area for future research is that of repeating the study in which the measurement process was modeled as the discrete system. In discrete case, the sampling time is very important. The theory of low sensitivity state feedback in the discrete case has not been investigated.

In addition, the study of the appropriateness of modeling the stochastic disturbances and sensor errors as white Gaussian processes, the effects of the propulsion system nonlinearities, the dynamics of slipping between the wheels and guideway for vehicles which are propelled by wheels, the relationships between the sensor spacings and sensor errors, the reaction time lags between the sensors and the control system are other possible research areas.

APPENDIX A

PROOF OF EQUATION (3.16) ON PAGE 3.7

It is desired to prove

$$\Phi(t,t_o) = \begin{bmatrix} \Phi_1(t,t_o) & 0 & 0 \\ \Phi_2(t,t_o) & \Phi_1(t,t_o) & 0 \\ \Phi_3(t,t_o) & 0 & \Phi_1(t,t_o) \end{bmatrix}$$

where

$$\Phi_1(t,t_o) = e^{(A-BG)(t-t_o)}$$

PROOF:

Since $\hat{A}$ is a constant matrix and $\Phi(t,t_o)$ is the fundamental matrix associated with (3.14), we have

$$\Phi(t,t_o) = e^{\hat{A}(t-t_o)}$$

$$= I_{3n} + \sum_{k=1}^{\infty} \frac{\hat{A}^k (t-t_o)^k}{k!} \qquad (A.1)$$

Firstly, the following claim is made

$$\hat{A}^k = \begin{bmatrix} (A-BG)^k & 0 & 0 \\ f_1(A-BG,A_w,k) & (A-BG)^k & 0 \\ f_2(A-BG,A_m,k) & 0 & (A-BG)^k \end{bmatrix} \qquad (A.2)$$

The claim (A.2) will be proved by the mathematical induction method. For k = 1,

$$\hat{A} = \begin{bmatrix} (A-BG) & 0 & 0 \\ A_w & (A-BG) & 0 \\ A_m & 0 & (A-BG) \end{bmatrix}$$

Suppose that $k = n$ is true in (A.2). Since $(A-BG)$, A_w, and A_m are of the same dimension, it may be calculated that

$$\hat{A}^{n+1} = \hat{A}\cdot\hat{A}^{n} = \begin{bmatrix} (A-BG)^{n+1} & 0 & 0 \\ f_1\cdot(A-BG)+(A-BG)^{n}A_w & (A-BG)^{n+1} & 0 \\ f_2\cdot(A-BG)+(A-BG)^{n}A_m & 0 & (A-BG)^{n+1} \end{bmatrix}$$

Thus, (A.2) holds true for $k = n+1$.

By plugging (A.2) into (A.1), we obtain

$$\Phi(t,t_o) =$$

$$\begin{bmatrix} I_n + \sum_{k=1}^{\infty} \frac{(A-BG)^k (t-t_o)^k}{k!} & 0 & 0 \\ \Phi_2(t,t_o) & I_n + \sum_{k=1}^{\infty} \frac{(A-BG)^k (t-t_o)^k}{k!} & 0 \\ \Phi_3(t,t_o) & 0 & I_n + \sum_{k=1}^{\infty} \frac{(A-BG)^k (t-t_o)^k}{k!} \end{bmatrix}$$

or

$$\Phi(t,t_o) = \begin{bmatrix} \Phi_1(t,t_o) & 0 & 0 \\ \Phi_2(t,t_o) & \Phi_1(t,t_o) & 0 \\ \Phi_3(t,t_o) & 0 & \Phi_1(t,t_o) \end{bmatrix}$$

where

$$\Phi_1(t,t_o) = I_n + \sum_{k=1}^{\infty} \frac{(A-BG)^k(t-t_o)^k}{k!}$$

$$= e^{(A-BG)(t-t_o)}$$

This completes the proof.

APPENDIX B

TRACE FUNCTION AND GRADIENT MATRIX

Let A be nxn matrix. Then the trace of A is defined as the sum of the elements on the main diagonal of A.

$$\mathrm{tr.}(A) \triangleq \sum_{i=1}^{n} a_{ii}$$

Properties of the trace of square matrices are

(B1) $\mathrm{tr.}(A) = \mathrm{tr.}(A^T) = \sum_{i=1}^{n} \lambda_i$ where the λ_i's are eigenvalues of A

(B2) $\mathrm{tr.}(AB) = \mathrm{tr.}(BA)$

(B3) $\mathrm{tr.}(kA) = k\,\mathrm{tr.}(A)$ where k is a constant

(B4) $\mathrm{tr.}(A+B) = \mathrm{tr.}(B+A) = \mathrm{tr.}(A)+\mathrm{tr.}(B)$

(B5) $\mathrm{tr.}(AX) = \mathrm{tr.}(XA) = x^TAx$ where $X = xx^T$

(B6) If $A = xy^T$, then $\mathrm{tr.}(A) = x^Ty$

(B7) $\mathrm{tr.}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \mathrm{tr.}(A) + \mathrm{tr.}(B)$

It is assumed that A, B, C, D, all have the same dimensions in (B1)-(B7).

Let X be a mxn real matrix and f(X) be a scalar-valued function of the elements x_{ij} of X. Then the gradient matrix of f(X) with respect to X, denoted by $\frac{\partial f(X)}{\partial X}$, is a mxn matrix defined by ([A1],[W3])

$$\frac{\partial f(X)}{\partial X} = \begin{bmatrix} \frac{\partial f(X)}{\partial x_{11}} & \frac{\partial f(X)}{\partial x_{12}} & \cdots\cdots & \frac{\partial f(X)}{\partial x_{1n}} \\ \vdots & & & \vdots \\ \frac{\partial f(X)}{\partial x_{m1}} & \frac{\partial f(X)}{\partial x_{m2}} & \cdots\cdots & \frac{\partial f(X)}{\partial x_{mn}} \end{bmatrix}$$

Defined $f(\cdot)$ to be a trace function of X if f(X) is of the form

$$f(X) = \text{tr.}[F(X)]$$

where $F(\cdot)$ is a continuously differential mapping from the space of mxn matrices into the space of nxm matrices. The trace function may be viewed as a particular class of scalar-valued function of a matrix. A method for calculating the gradient matrices for trace functions based on the theory of perturbation is developed in Ref. [K1]. The procedures are given here for completeness.

Step 1: Calculate

$$f(X+\epsilon\Delta X)-f(X) = \text{tr.}[F(X+\epsilon\Delta X)-F(X)]$$

Step 2: Expand $F(X+\epsilon\Delta X)$ in terms of ϵ and get a first order term in ϵ , i.e.,

$$F(X+\epsilon\Delta X) = F(X) + \epsilon L(\Delta X)$$

where $L(\Delta X)$ is a continuous linear mapping from the space of nxm matrices into the space of nxn matrices.

Step 3: Using formula (B1)-(B4), write

$$\text{tr.}[F(X+\epsilon\Delta X)-F(X)] = \epsilon\ \text{tr.}[L(\Delta X)]$$

in the form of

$$\text{tr.}[F(X+\epsilon\Delta X)-F(X)] = \epsilon\ \text{tr.}[M(X)\Delta X]$$

where $M(X)$ is a nxm matrix.

Step 4: Then the gradient matrix is given by

$$\frac{\partial f(X)}{\partial X} = M^T(X)$$

Using these procedures, the following formula needed in Chapter 3 and 4 may be obtained.

(B8) $\frac{\partial}{\partial X}\ \text{tr.}(X) = I_n$

(B9) $\frac{\partial}{\partial X}\ \text{tr.}(AX) = A^T$

(B10) $\frac{\partial}{\partial X}\ \text{tr.}(AX^T) = A$

(B11) $\frac{\partial}{\partial X}\ \text{tr.}(AXB) = \frac{\partial}{\partial X}\ \text{tr.}(XBA) = A^TB^T$

(B12) $\frac{\partial}{\partial X}\ \text{tr.}(AX^TB) = \frac{\partial}{\partial X}\ \text{tr.}(X^TBA) = BA$

(B13) $\frac{\partial}{\partial X}\ \text{tr.}(X^TAX) = (A+A^T)X$

(B14) $\frac{\partial}{\partial X}\ \text{tr.}(B^TX^TAXB) = (A+A^T)XBB^T$

(B15) $\frac{\partial}{\partial X}\ \text{tr.}(B^TC^TX^TAXCB) = (A+A^T)XCBB^TC^T$

From (B8) to (B15) it is assumed that the elements x_{ij} of X are mutually independent.

APPENDIX C

FLOW CHARTS OF COMPUTER PROGRAMS

The following flow charts of digital computer programs were used to obtain the simulation results in Chapter 5. The derivation of the iterative algorithm for solving the matrix Lyapunov equation can be found in Ref.[K3]. The complete flow charts of computer programs used in Chapter 5 are listed below.

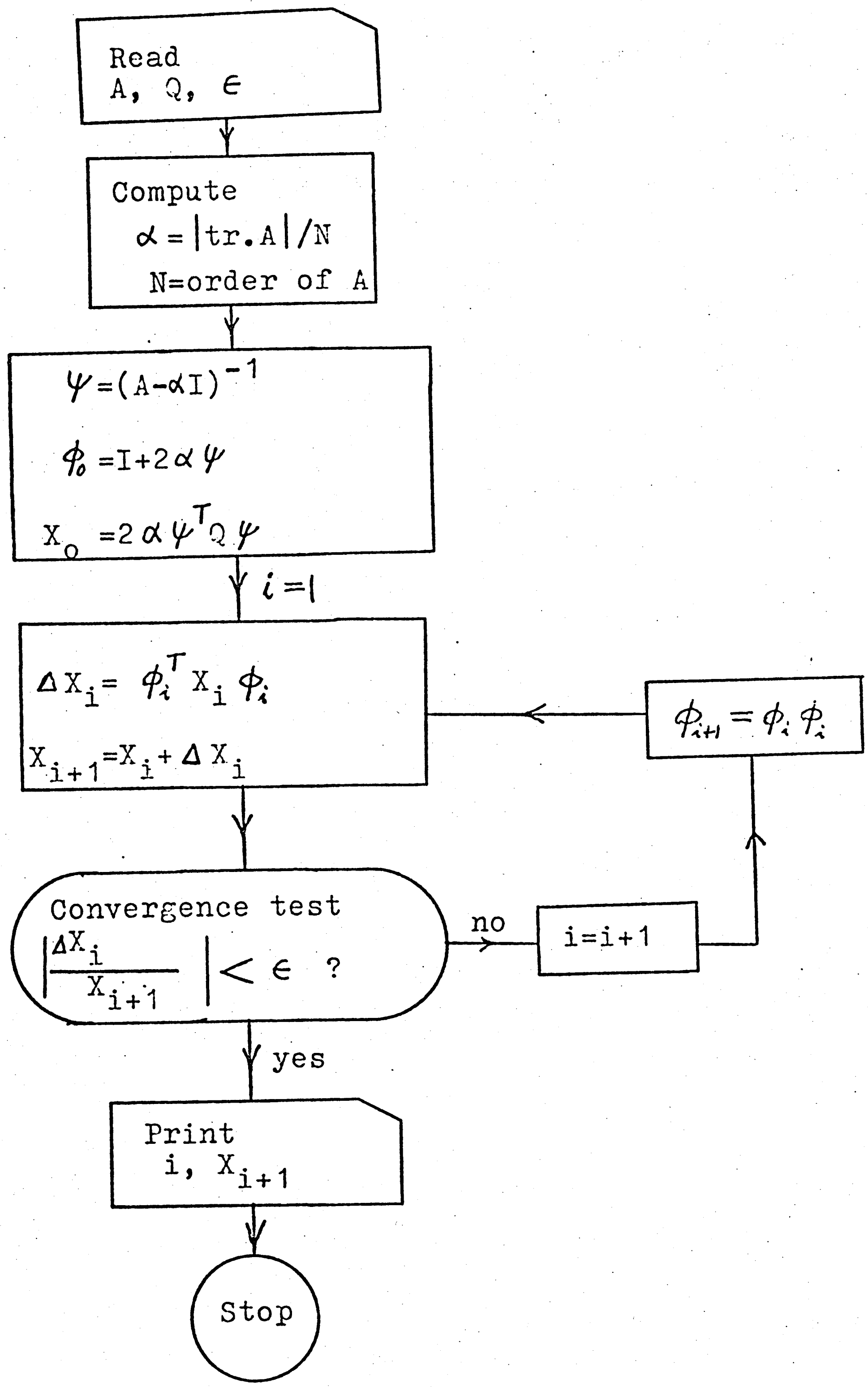


Fig. C.1 Flow chart of the iteration algorithm for solving the matrix Lyapunov equation $XA+A^TX+Q=0$

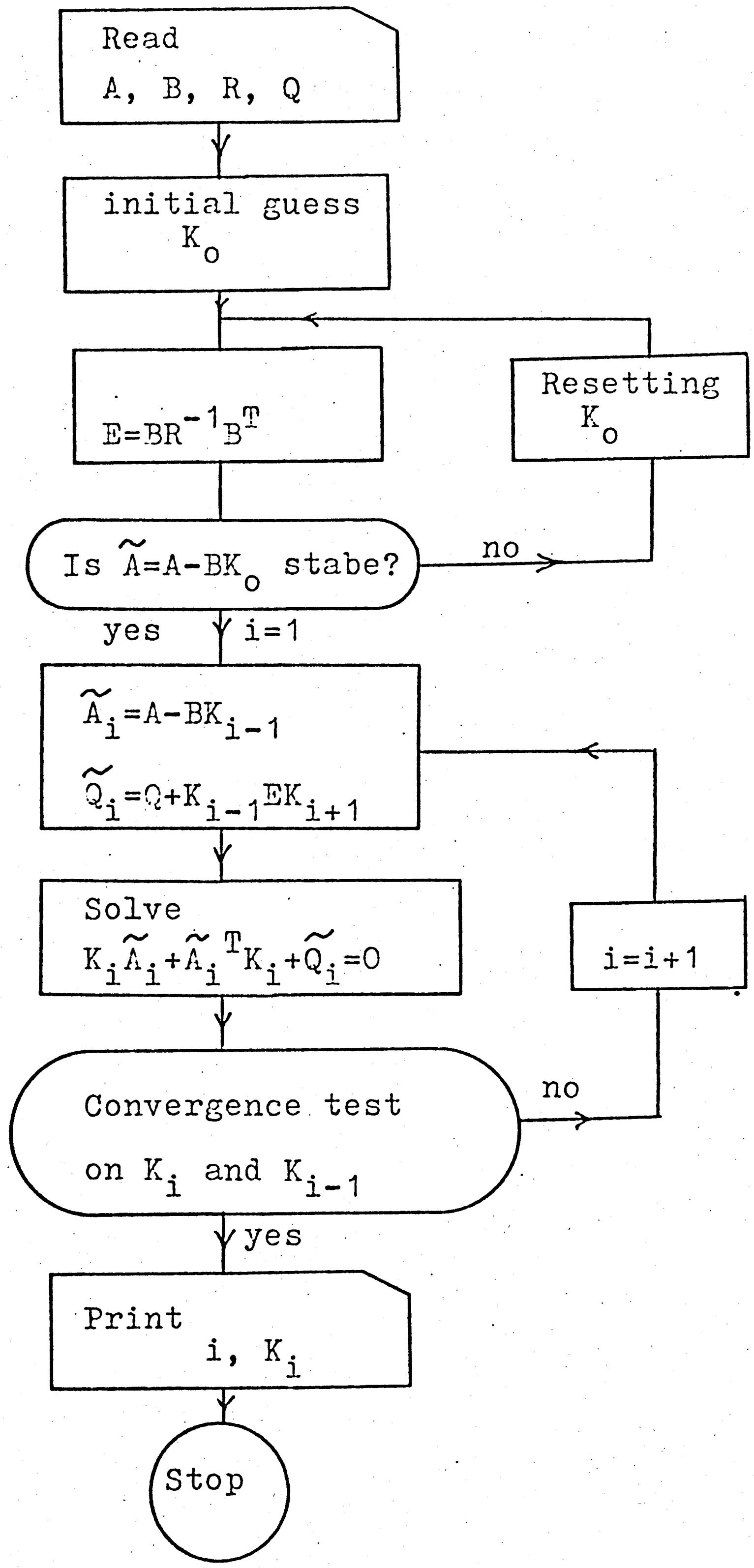


Fig. C.2 Flow chart of the iterative algorithm for solving the matrix Riccati equation $KA+A^TK-KBR^{-1}B^TK+Q=0$

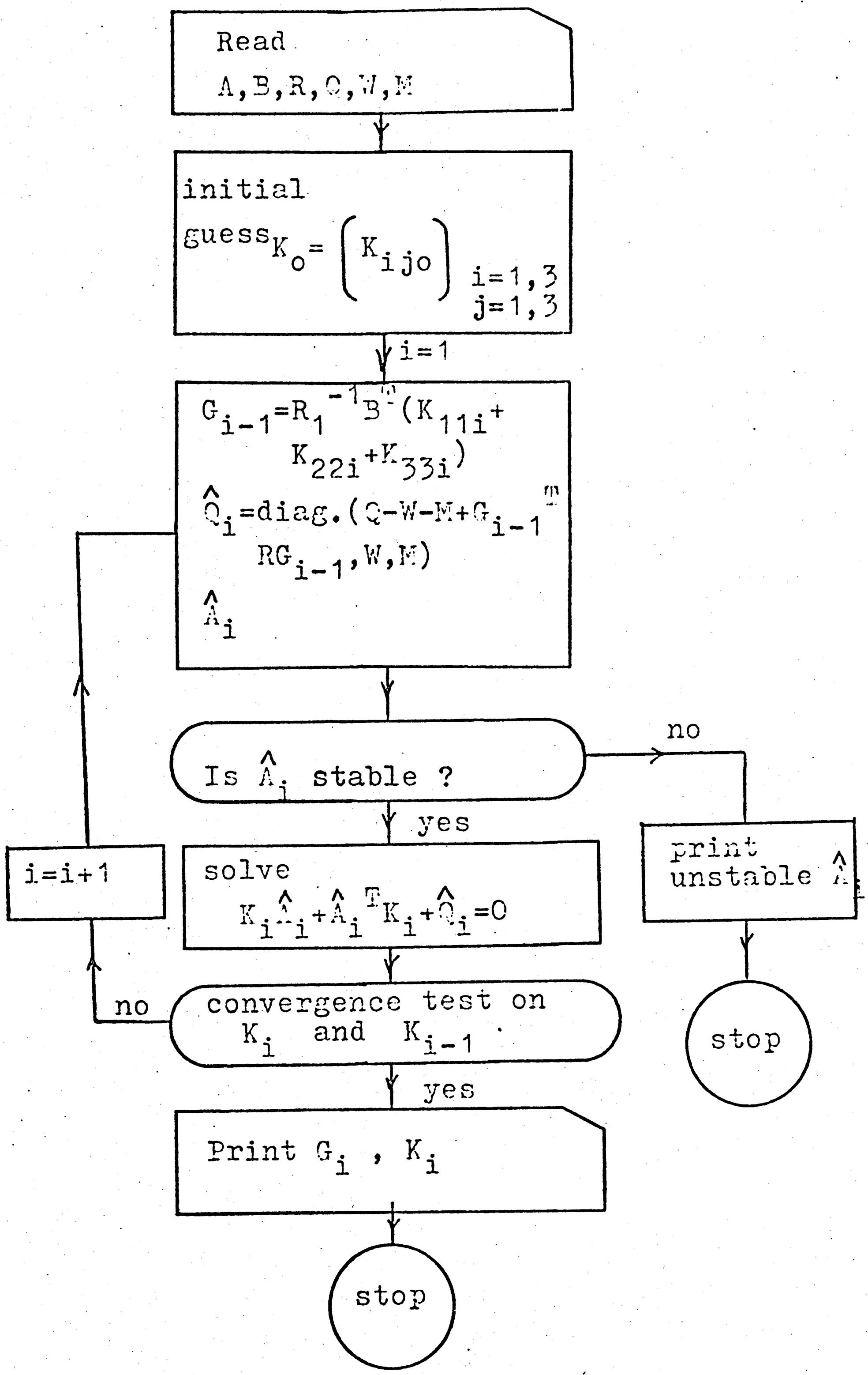


Fig.C.3 Flow chart of the iterative algorithm for solving the low sensitivity gain matrix

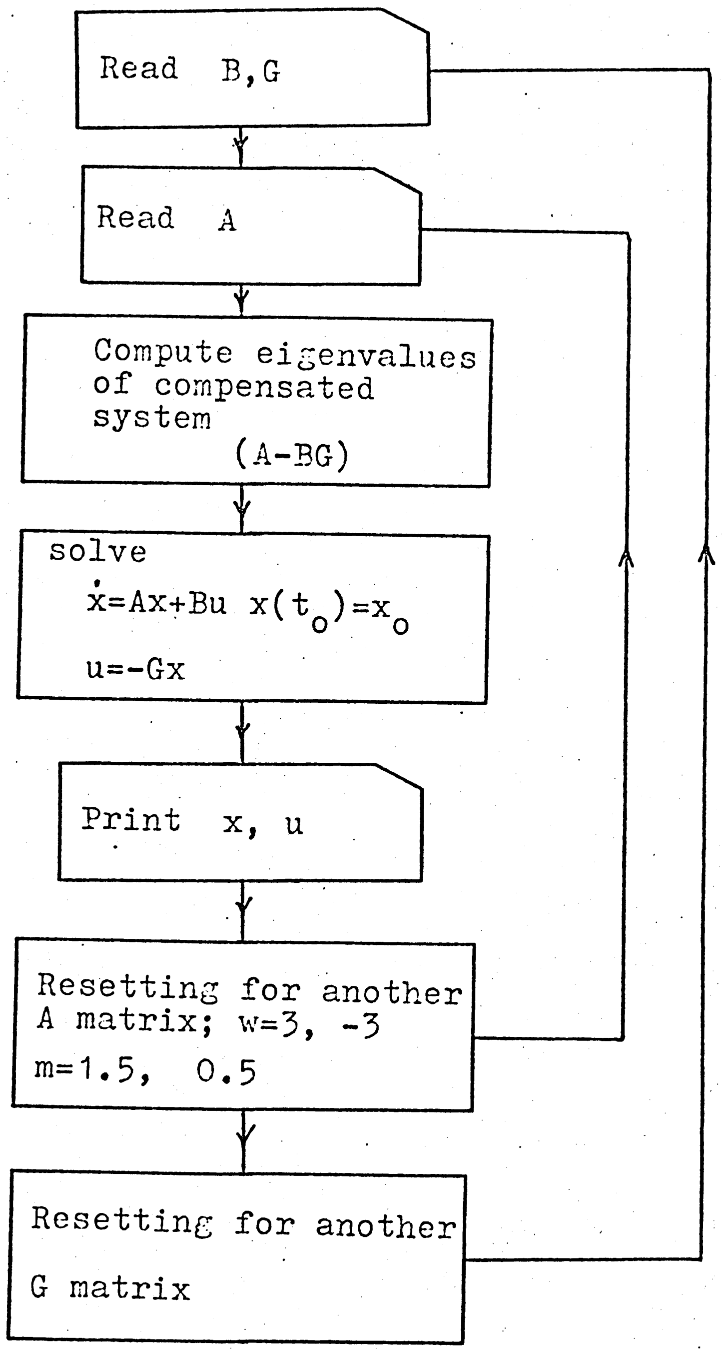


Fig.C.4 Flow chart of the mainline normal operation

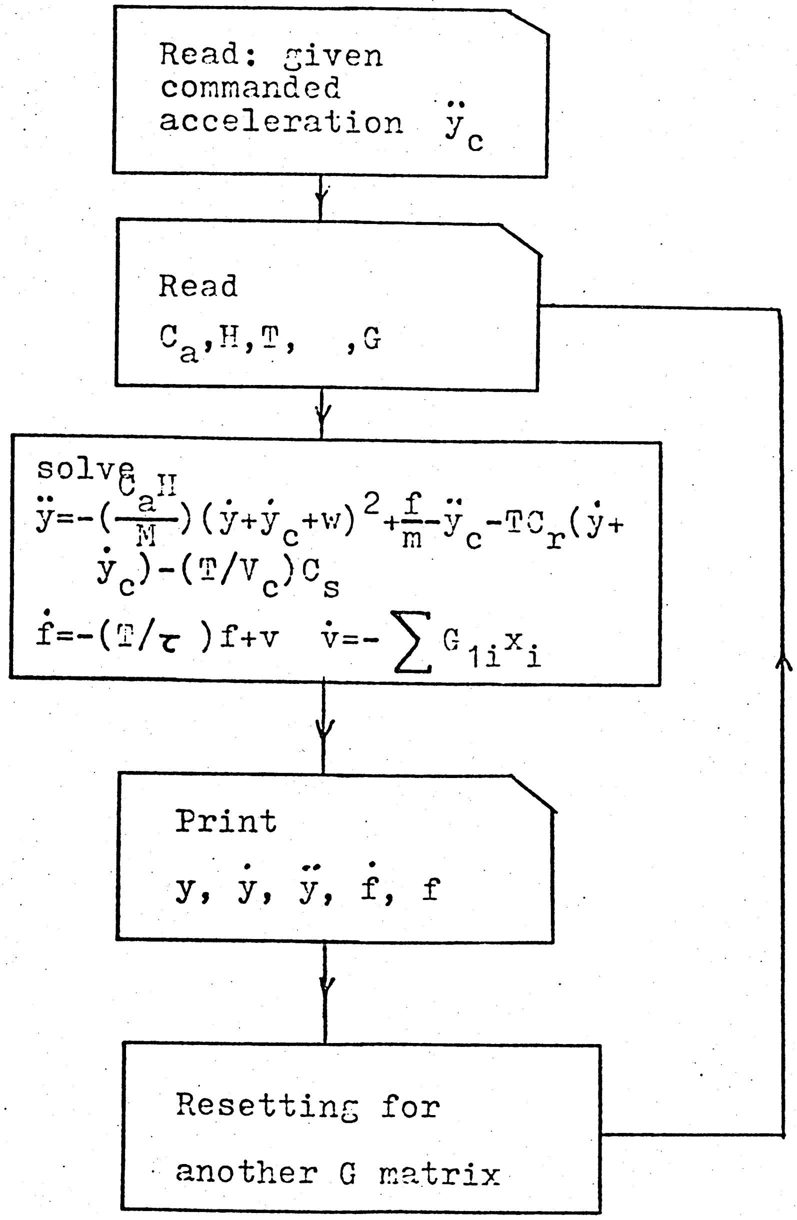


Fig.C.5 Flow chart of the nonlinear profile-following operations

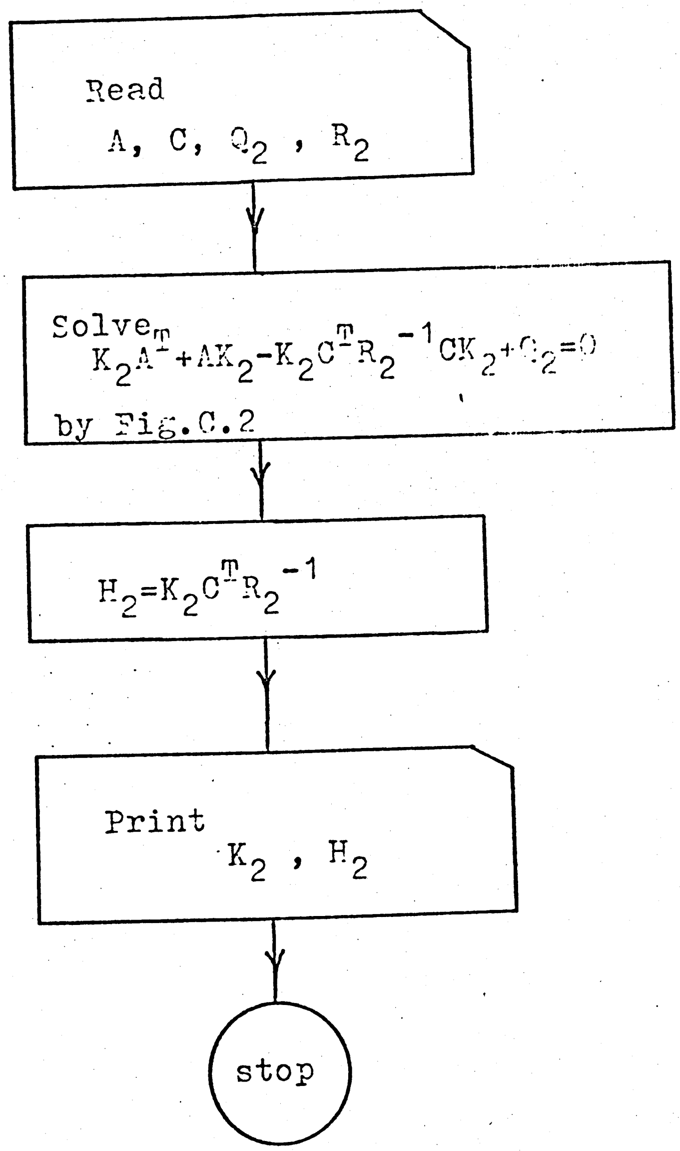


Fig.C.6 Flow chart for computation of optimal filtering gain matrix H_2

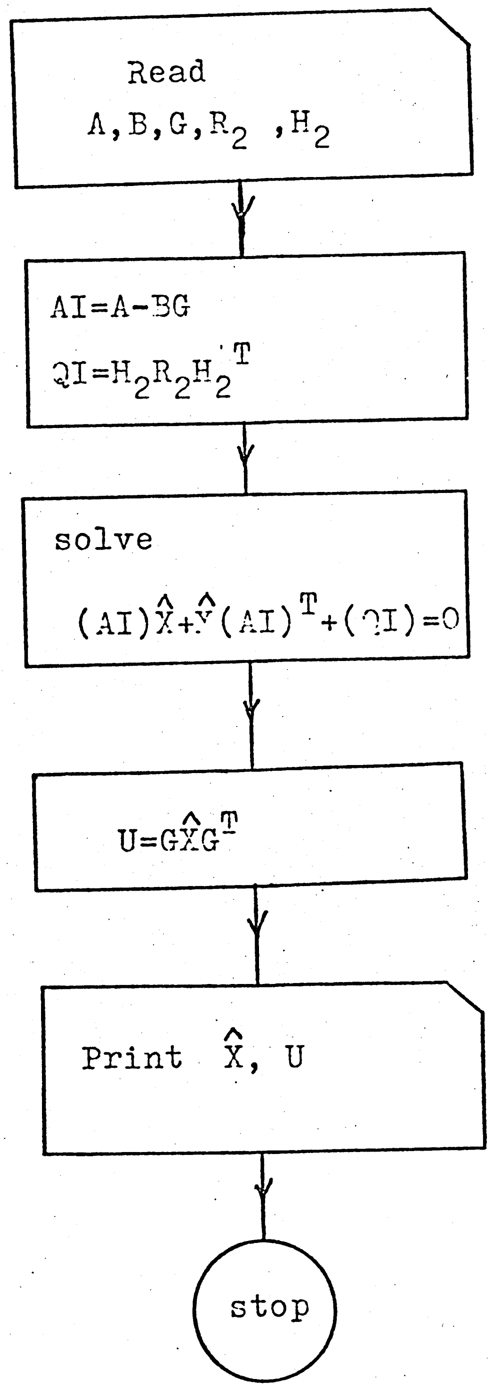


Fig.C.7 Flow chart for calculation of steady state behavior in presence of input disturbance and sensor errors

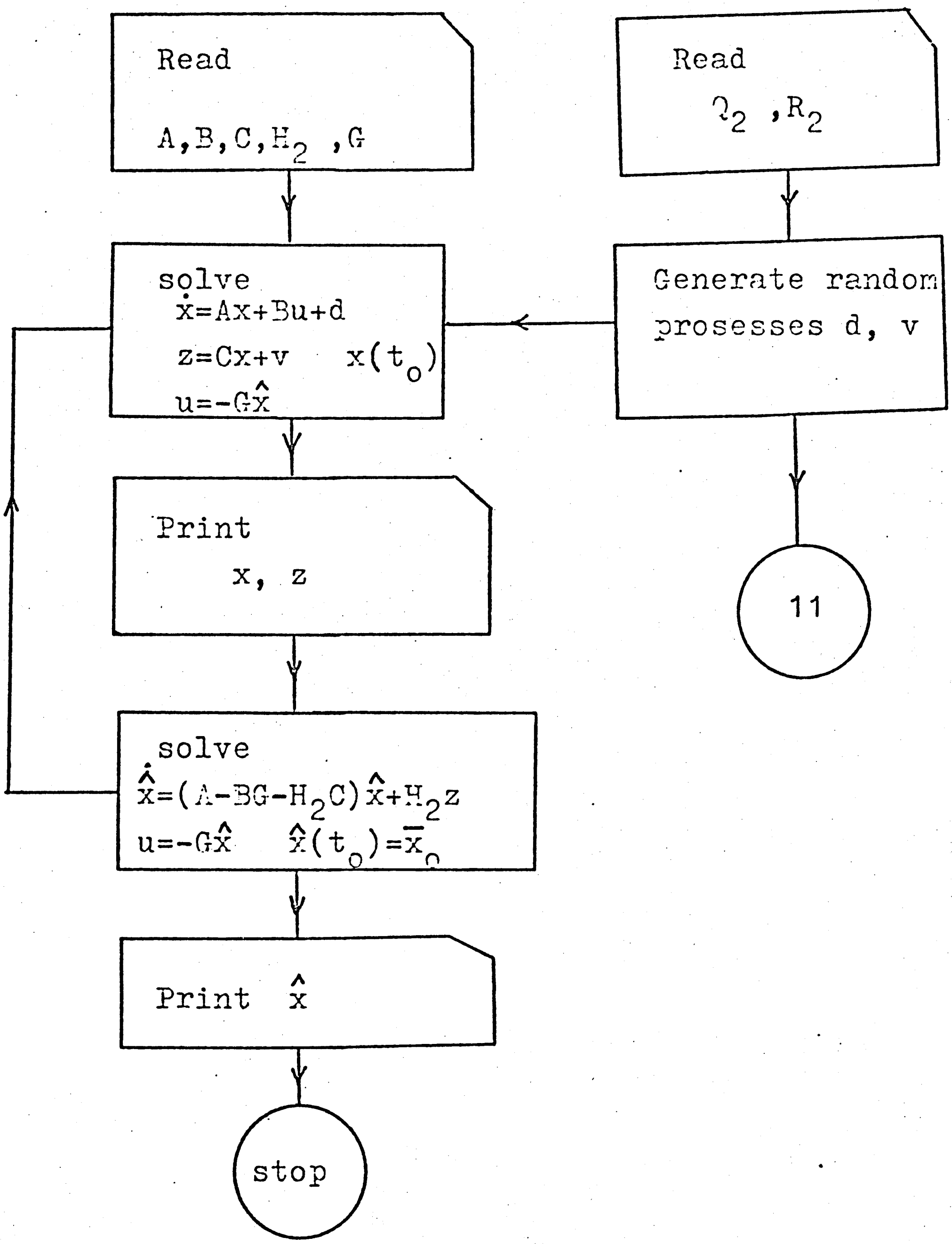


Fig.C.8 Flow chart for the stochastic vehicle dynamic response

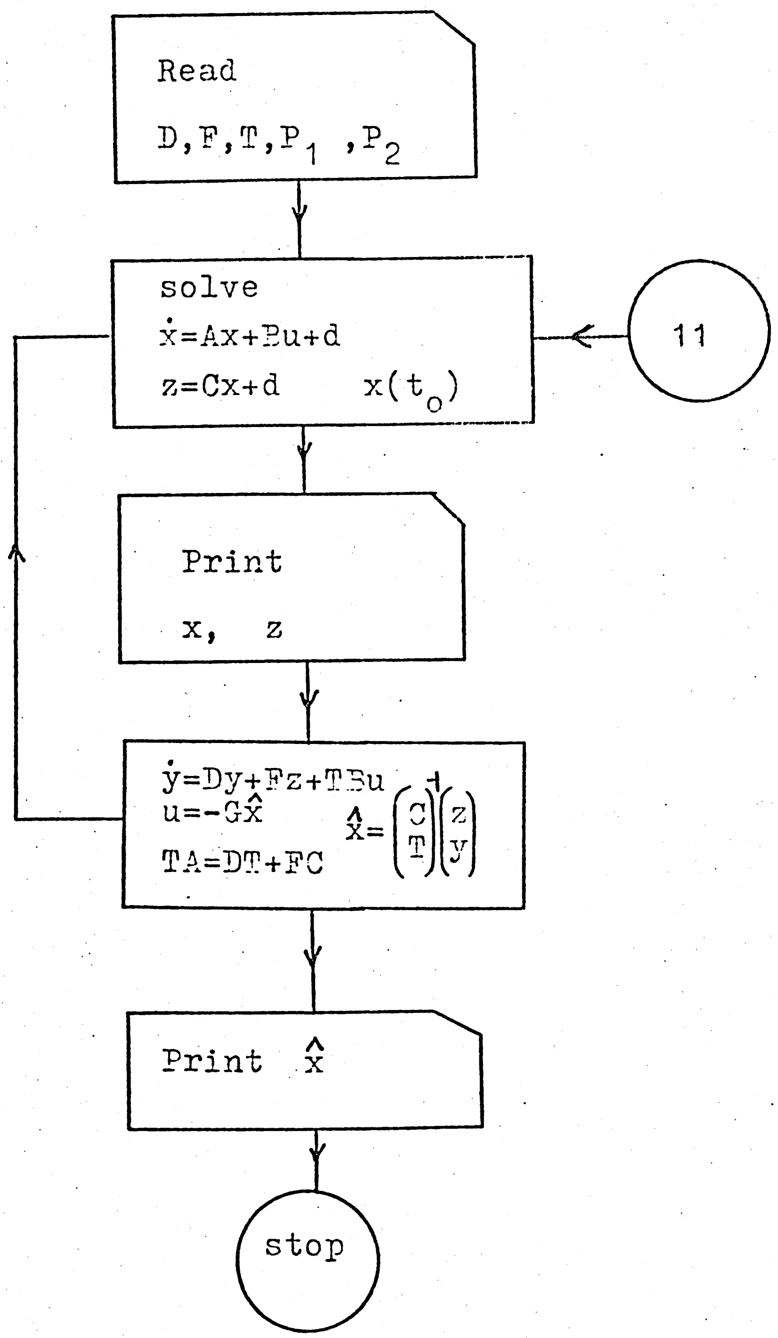


Fig.C.8(continued from previous page)

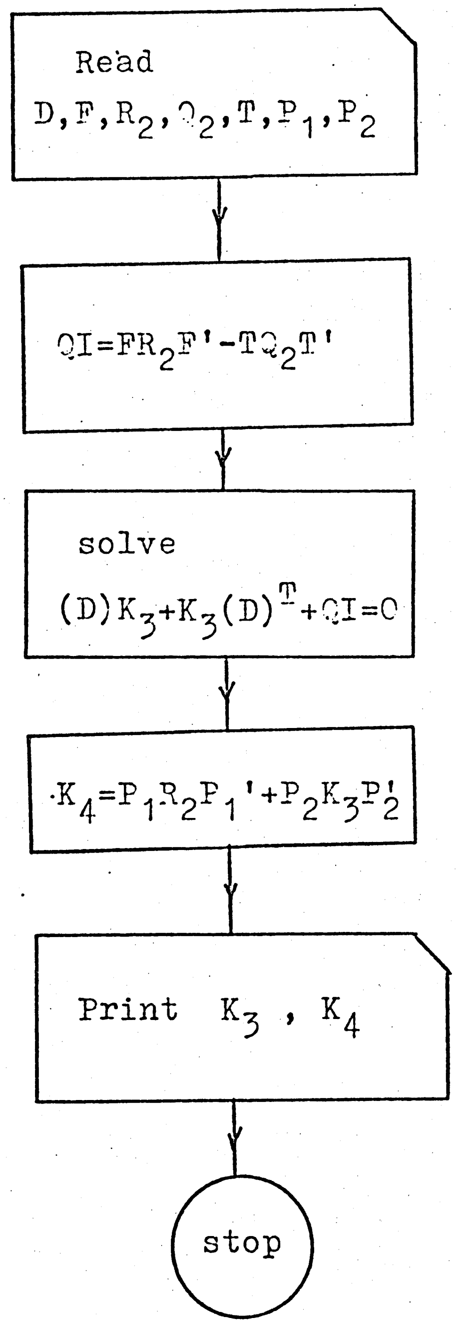


Fig.C.9 Flow chart for calculation of steady state behavior for observers

APPENDIX D

DERIVATION OF STEADY-STATE BEHAVIOR OF THE OBSERVER

An observer is a dynamic system which operates on the output of the original system to reconstruct the missing state variables of the original system. ([L7]), [G4]). The steady-state behavior of the observer in the presence in input disturbances and sensor errors will be derived in the below.

Consider the completely observable system described by

$$\dot{x} = Ax + Bu + d \tag{D.1}$$

$$z = Cx + v \tag{D.2}$$

where x is the state of the original system

z is the measured output of the original system

u is the control input applied to the system

d is the white Gaussian disturbances with zero mean and constant covariance matrix Q_2

v is the white Gaussian sensor errors with zero mean and constant covariance matrix R_2.

The state of the observer is notated as y where y satisfies the following observer equation ([L7],[L8])

$$y = Tz + \varepsilon \tag{D.3}$$

and

$$\dot{y} = Dy + Fz + TBu \tag{D.4}$$

with

$$TA = DT + FC \tag{D.5}$$

where ε is the observer error.

The estimate of the state of the original system is given by

$$\hat{x} = \begin{bmatrix} C \\ T \end{bmatrix}^{-1} \begin{bmatrix} z \\ y \end{bmatrix} = [P_1 \quad P_2] \begin{bmatrix} z \\ y \end{bmatrix} \tag{D.6}$$

where the existence of the matrix inverse is assumed and denoted by

$$\begin{bmatrix} C \\ T \end{bmatrix}^{-1} = [P_1 \quad P_2] \tag{D.7}$$

The equation (D.5) may be written as

$$TA = [F \quad D] \begin{bmatrix} C \\ T \end{bmatrix} \tag{D.8}$$

Using the equation (D.7), the above relation yields

$$F = TAP_1 \tag{D.9}$$

$$D = TAP_2 \tag{D.10}$$

From equations (D.1)-(D.4), the observer error ε can be shown to satisfy the following differential equation

$$\dot{\varepsilon} = D\varepsilon + Fv - Td \tag{D.11}$$

The steady-state behavior of the observer error is obtained from (D.11) as

$$0 = DK_3 + K_3D' + FR_2F' - TQ_2T' \tag{D.12}$$

where $K_3 = E[\varepsilon\varepsilon']$ is the covariance of the observer error.

Since the observer output and the state of the original system is related by (D.2) and (D.4),

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} C \\ T \end{bmatrix} x + \begin{bmatrix} v \\ \varepsilon \end{bmatrix} \tag{D.13}$$

Combining (D.7) and (D.13) with (D.6), we get

$$\hat{x} = x + [P_1 \quad P_2] \begin{bmatrix} v \\ \varepsilon \end{bmatrix} \tag{D.14}$$

The resulting estimation error is

$$e \triangleq x - \hat{x} = -(P_1v + P_2\varepsilon) \tag{D.15}$$

Finally, the covariance matrix of the estimation error is obtained from (D.15) as

$$K_4 = E[ee'] = P_1R_2P_1' + P_2K_3P_2' \tag{D.16}$$

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